TEORIA DELLE PIASTRE - Prof. Marco Lemmo - 6 CFU - A.A. 2008/09

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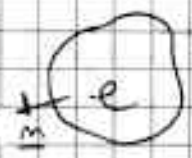
TEORIA DELLE PIASTRE - Prof. Lenero Tornio 2/3/09

H

Conosciamo le eq. indef. di equilibrio



$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

FD alle DEF. PAR.

Cond. al contorno: spostamento $u = \hat{u}$, $v = \hat{v}$, $w = \hat{w}$ o spinte tensione $t_{nx} = \hat{t}_{nx}$, $t_{ny} = \hat{t}_{ny}$, $t_{nz} = \hat{t}_{nz}$ [cond. alternative, O - O].

Sist. estrem. complicato nel caso generale ⇒ necessità teoria delle TRAVE, + semplice.

DS. UFNSNT elementi monodimensionali ad asse rettilineo
Eq. alle derivate ordinarie.

MA non tutti i corpi sono travi così.
E " " " " " continui.
Travi: 1 dim. prevalente sulle altre
∃ cilindri con altezza piccola risp. alle dim. delle basi.

→ PIASTRA

In trave si riconducono eq. su f. def. su z. (da 3d a 1d)
In piastra si individua il PIANO MEDIO.
∃ <> teorie da 3d a 1d. FQ alle D.P.
Non ci sono soluzioni generali. Interviene il calcolo numerico ↳ (cioè sol. di casi semplici noti o sol. analitiche) con gli elementi finiti:
- Minimo del funz. dell'energia del probl.

di equilibrio

- formulaz. variaz.

Metodo approssimato ricerca soluzioni ⇒ metodo FEM.

Th. piastre è in evoluzione. ∃ teorie classiche da oltre 100 anni, e altre analitiche - computazionali. Argomento attuale. In ing. civile ex. solai a soletta piena pareti serbatoi, ...

Attualità in materiali COMPOSITI (fibre in matrice), ricerca, applicaz.

Non tutti i corpi sono rettilinei, o cilindrici. ∃ i GUSCI che non hanno una sup. media piana.

A

PRELIMINARI DI CALCOLO E ALGEBRA

- <u>Spazio vettoriale lineare</u> $\mathcal{L}$

Ins. di elementi dove si può definire la SOMMA e il PRODOTTO con certe proprietà.

- $\underline{a} + \underline{b} = \underline{c} \in \mathcal{L}$ ovviamente (con $\underline{a}, \underline{b} \in \mathcal{L}$)
  - > ∃ elem. NULLO $\underline{a} + \underline{0} = \underline{a} \quad \forall \underline{a} \in \mathcal{L}$
  - > ∃ " OPPOSTO $\underline{a} + (-\underline{a}) = \underline{0}$
  - > ∃ prop. COMMUTATIVA $\underline{a} + \underline{b} = \underline{b} + \underline{a}$
  - > ∃ " ASSOCIATIVA $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$

- $\lambda \underline{a} = \underline{b}$
  - > $1 \cdot \underline{a} = \underline{a} \quad \forall \underline{a} \in \mathcal{L}$
  - > $\lambda(\mu \underline{a}) = (\lambda\mu)\underline{a}$ (assoc.)
  - 2 > $\lambda(\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$

> $(\lambda+\mu)\,\underline{a} = \lambda\,\underline{a} + \mu\,\underline{a}$

Le 10 propr. sono gli ASSIOMI.

Esempi su $\mathcal{L}$ sono: $\mathbb{R}$ $(x+y\,;\ \lambda x)$, $\mathbb{R}^n$ con $(x_1, \dots, x_n)$ e $(y_1, \dots, y_n)$ dove $(x+y = x_1+y_1, \dots, x_n+y_n)$ e $(\lambda x = \lambda x_1, \dots, \lambda x_n)$.

(Vedi n. 2 dispensa)

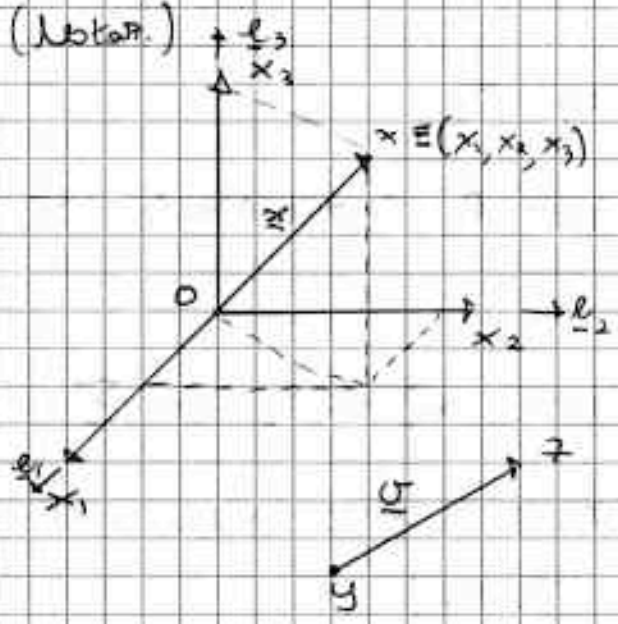
- V $\quad \underline{u} + \underline{v} = \underline{w}$, $\lambda \underline{u}$
- L_{IN} (trasf. lineari)

Negli $\mathcal{L}$ si può definire il PRODOTTO SCALARE

$\mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{R}$ (spazio EUCLIDEO) spazio globale di prod. scalare

- $\underline{a} \cdot \underline{a} \geq 0$ se $\underline{a} \cdot \underline{a} \iff \underline{a} = \underline{0}$
- $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ (commutativa)
- $(\lambda \underline{a}) \cdot \underline{b} = \lambda (\underline{a} \cdot \underline{b})$
- $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$

(Notaz.)



$x, y, z \dots$ i vettori in grassetto nello stampato, sottolineato su carta.

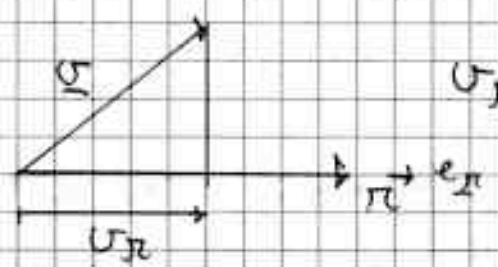
Corr. biunivoca tra punti e coord. cartesiane.

Vettore segm. orientato che unisce 2 punti, espriam. come differenza tra 2 p.

$\underline{v} = z - y$.

Somma di punto e vettore è punto. $y + \underline{v} = z$

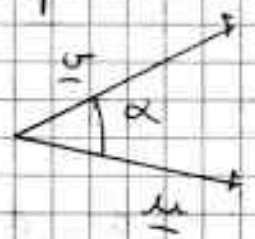
Non ho significato la somma di 2 punti.
Spazio 3d è EUCLIDEO + $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$
Chiamato con E
Corr. $x \in E$. Corrisponde il VETTORE POSIZIONE $\underline{x}$
Un vettore si può esprimere nelle comp. cartesiane $\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$



$v_\pi = \underline{v} \cdot \underline{e}_\pi$. $\underline{u} \cdot \underline{v} = u v \cos\alpha$
↓ quindi

$\underline{v}_1 = \underline{v} \cdot \underline{e}_1$; $\underline{v}_2 = \underline{v} \cdot \underline{e}_2$; $\underline{v}_3 = \underline{v} \cdot \underline{e}_3$

$\underline{v} \equiv (v_1, v_2, v_3)$, $\underline{x} \equiv (x_1, x_2, x_3)$ cioè le coordinate del punto x. Quindi stabiliamo corr. biun tra • e vett. posiz. $x \longleftrightarrow \underline{x}$, comodo confondere.



$\underline{u} \cdot \underline{v} = u v \cos\alpha$

$\underline{u} = \underline{u}_1 e_1 + \underline{u}_2 e_2 + \underline{u}_3 e_3$

$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Comodo usare gli indici per le coordinate x_i.
Quindi:

$\underline{u} \cdot \underline{v} = \sum_{i=1}^{3} u_i v_i$. Conviene abolire la Σ e usiamo la CONVENZIONE DELLA SOMMA SUGLI INDICI RIPETUTI: $= u_i \cdot v_i$ (con i nota).
Allora $\underline{u} = u_h \underline{e}_h$ (h, i non importa, basta che sia ripetuta) [NOTAZIONE INDICIALE, se uso $\underline{u} \cdot \underline{v}$ ho la NOTAZIONE DIRETTA]

Si usa il DELTA DI KRONECKER

$$\delta_{ij} = \begin{cases} 1 & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases} \quad ; \quad \delta_{ij} = \underline{e}_i \cdot \underline{e}_j$$

Usato nel prod. scalare:

$$\underline{u} \cdot \underline{v} = u_i \, e_i \cdot v_m \, e_m = u_i \, v_m \, \underline{e}_i \cdot \underline{e}_m$$

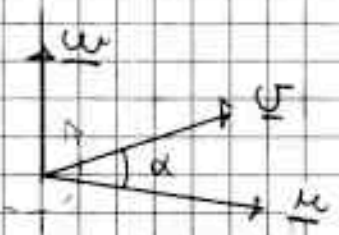
(grazie alla prop. distributiva) =

$$= u_i \, v_m \, \delta_{im} = u_i \, v_i \quad \text{CVD.}$$

$$\underline{u} = \sqrt{\underline{u} \cdot \underline{u}} = \sqrt{u_k u_k} = \sqrt{u_1 u_1 + u_2 u_2 + u_3 u_3} =$$

$$= \sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2} \quad \text{(MODULO).}$$

PRODOTTO VETTORIALE



$$\underline{u} \times \underline{v} = \underline{w}$$

Modulo $w = u v \sin\alpha$

Nullo se $\underline{u}$ o $\underline{v} = \underline{0}$ o se $\underline{u} \parallel \underline{v}$.

Si usa il SIMBOLO DI GREGORIO RICCI, geometra italiano

$$\varepsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k$$

Possiamo noi fare $(\underline{u} \times \underline{v}) \cdot \underline{w}$ (prodotto misto)

$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$

$\underline{e}_2 \times \underline{e}_1 = -\underline{e}_3$ (p.u. e alternante)

$\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$

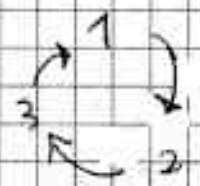
$\underline{e}_3 \times \underline{e}_1 = \underline{e}_2$

Allora si ha che

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{se due indici sono uguali} \\ 1 & \text{se } (i,j,k) \text{ sono permut. pari } (1,2,3) \\ -1 & \text{se } (i,j,k) \quad '' \quad '' \quad \text{dispari} \end{cases}$$

Ex: $\underline{e}_1 \times \underline{e}_3 = \underline{e}_3$

Permut. pari: M. pari di scambi $(i=3,\ j=1,\ k=2)$ ottenuta da $(2,3,1) \to (3,1,2) \to (1,2,3)$. O meglio:



Per $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$. Enum cambia con permut. ciclica indici.

$$\underline{u} \times \underline{v} = u_i \underline{e}_i \times v_j \underline{e}_j =$$

[Cons $\underline{w} = w_m \underline{e}_m = (\underline{w} \cdot \underline{e}_m)\underline{e}_m$]

$$= \Big(\underbrace{[u_i \underline{e}_i \times v_j \underline{e}_j]}_{\underline{w}} \cdot \underline{e}_k\Big)\underline{e}_k = \varepsilon_{ijk} u_i v_j \underline{e}_k$$

$$(\underline{u} \times \underline{v})_1 = u_2 v_3 - u_3 v_2$$

$$(\underline{u} \times \underline{v})_2 = u_3 v_1 - u_1 v_3$$

$$(\underline{u} \times \underline{v})_3 = u_1 v_2 - u_2 v_1$$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(Laplace con P.S, Ricci con P.V).

Si può fare $(\underline{u} \times \underline{v}) \times \underline{w} = \varepsilon_{ijk} (\underline{u} \times \underline{v})_i w_j \underline{e}_k =$

$$= \varepsilon_{ijk} \varepsilon_{pqi} u_p v_q w_j \underline{e}_k$$

Risultato in mat. diretta.

(6) $(\underline{u} \times \underline{v}) \times \underline{w} = \underline{v}(\underline{u} \cdot \underline{w}) - \underline{u}(\underline{v} \cdot \underline{w})$ [Risultato sarà sempre $\perp$ a $\underline{w}$]

Consr. insieme vettori spazio E bidim. V.
Consr. l'applicazione $A: V \xrightarrow{lin} V$
$\underline{A}: \underline{u} \in V \longmapsto \underline{A}(\underline{u}) = \underline{v} \in V$
$\underline{A}$ è lineare significa: Consr l'azione $\underline{A}(\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2) = \alpha_1 \underline{A}\, \underline{u}_1 + \alpha_2 \underline{A}\, \underline{u}_2$
Consr. TENSORI ([se non spec.] DEL SECONDO ORDINE)
LIN: insieme delle trasf lineari nello spazio delle V
L'insieme di tutti i LIN è uno sp. vett.
$(\underline{A} + \underline{B})\, \underline{u} = \underline{A}\, \underline{u} + \underline{B}\, \underline{u}$
$(\lambda \underline{A})\, \underline{u} = \lambda (\underline{A}\, \underline{u})$

3/3/2009

Lin $\underline{A}: U \xrightarrow{LIN} U$
$\underline{A}: u \in U \longmapsto \underline{A}\, \underline{u} = \underline{v} \in U$

$A_{ij} = (\underline{A}\, \underline{e}_j) \cdot \underline{e}_i$. Scrivo vett. in componenti.
$\underline{A}\, \underline{e}_j \cdot \underline{e}_i = (\underline{A}\, \underline{e}_j)_i$. Ogni vettore è $\underline{u} = u_k \underline{e}_k$
$\underline{A}\, \underline{e}_j = (\underline{A}\, \underline{e}_j)_i\, \underline{e}_i = A_{ij}\, \underline{e}_i$
Posso scrivere $\underline{A}\, \underline{e}_j = A_{ij}\, \underline{e}_i$
$\underline{A}\, \underline{e}_1 = A_{i1}\, \underline{e}_i$
Applicando A ottengo: $\underline{v} = \underline{A}\, \underline{u}$
$\underline{v} \cdot \underline{e}_i = \underline{A}\, u \cdot \underline{e}_i$. Possiamo esprimere $\underline{u} = u_j \underline{e}_j$
$= \underline{A}(u_j \underline{e}_j)\, \underline{e}_i = u_j (\underline{A}\, \underline{e}_j) \cdot \underline{e}_i$

[Porto scalare fuori essendo A trasf. lineare.]

$= u_j A_{kj} \underline{e}_k \cdot \underline{e}_i = A_{ij} u_j = [\underline{v} \cdot \underline{e}_i]$

$i = 1,2,3$. Ex, se $i=3$, $v_3 = A_{31}u_1 + A_{32}u_2 + A_{33}u_3$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Siamo partiti da $\underline{v} = \underline{A}\,\underline{u}$ e usando l'espress. in componenti abbiamo ottenuto matrice.

- <u>Composiz.</u> di trasf. lineari.

$\underline{A}(\underline{B}\,\underline{u}) = \underline{C}\,\underline{u}$ se $\underline{C} = \underline{A}\,\underline{B}$

$C_{ij} = \left(\underline{A}(\underline{B}\,\underline{e}_j)\right) \cdot \underline{e}_i = \left(\underline{A}\, B_{kj}\, \underline{e}_k\right) \cdot \underline{e}_i =$

$= \left(A_{mk}\, \underline{e}_m\, B_{kj}\right) \cdot \underline{e}_i = A_{ik}\, B_{kj}$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{bmatrix}$$

Stessa regola di comp. delle matrici.

- <u>Trasposto</u> del tensore $\underline{A}$.

Possiamo porre $\underline{u} \cdot \underline{A}\,\underline{v} = \underline{A}^T \underline{u} \cdot \underline{v} \quad \forall\, u, v \in V$

(trasf. lineare su altro vettore prendendo il trasposto).

$(\underline{A}^T)_{ij} = A_{ji}$. Infatti $(\underline{A}^T)_{ij} = \underline{A}^T \underline{e}_j \cdot \underline{e}_i =$

$= \underline{e}_j \cdot \underline{A}\,\underline{e}_i = A_{ji}$

- Tensore è simmetrico se $\underline{S} = \underline{S}^T$: sym.

$S_{ij} = S_{ji}$

- Tensore è antisimmetrico se $\underline{W} = -W^T$

$W_{ij} = -W_{ji}$

Un tensore sym ha 6 comp. distinti.
($S_{12} = S_{21}$, ecc.)
In un tensore skw le comp. sulle diagonali sono nulle e quelle fuori sono opposte, quindi 3 distinti componenti.

- $\underline{A} = \text{sym}\,\underline{A} + \text{skw}\,\underline{A}$ avendo posto:

$\text{sym}\,\underline{A} = \frac{1}{2}(\underline{A} + \underline{A}^T)$, $\text{skm}\,\underline{A} = \frac{1}{2}(\underline{A} - \underline{A}^T)$

- $(\underline{B} + \underline{C})^T = \underline{B}^T + \underline{C}^T$. Dim:

$\underline{u} \cdot (\underline{B} + \underline{C})\underline{v} = \underline{u} \cdot (\underline{B}\underline{v} + \underline{C}\underline{v}) = \underline{u} \cdot \underline{B}\underline{v} + \underline{u} \cdot \underline{C}\underline{v} =$

$= \underline{B}^T \underline{u} \cdot \underline{v} + \underline{C}^T \underline{u} \cdot \underline{v} = (\underline{B}^T + \underline{C}^T)\underline{u} \cdot \underline{v}$ cvd.

- $(\text{sym}\,\underline{A})^T = \frac{1}{2}(\underline{A}^T + \underline{A}) = \text{sym}\,\underline{A}$
- $(\text{skw}\,\underline{A})^T = \frac{1}{2}(\underline{A}^T - \underline{A}) = -\text{skw}\,\underline{A}$

In componenti: $(\text{sym}\,\underline{A})_{ij} = \frac{1}{2}(\underline{A}\,\underline{e}_j \cdot \underline{e}_i + \underline{A}^T \underline{e}_j \cdot \underline{e}_i) =$

$= \frac{1}{2}(A_{ij} + A_{ji})$

$(\text{skw}\,\underline{A})_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$

(8) Ogni elemento del Lin è quindi somma di

tens sym e tens. antisym (skw).

Sym

$$Lin = Sym \oplus Skw$$

Skw

- Trasposto del composto

$$(\underline{B}\,\underline{C})^T = \underline{C}^T\,\underline{B}^T \qquad \text{Dim:}$$

$$\underline{u}\cdot\underline{B}(\underline{C}\,\underline{v}) = \underline{B}^T\underline{u}\cdot\underline{C}\,\underline{v} = \underline{C}^T\underline{B}^T\underline{u}\cdot\underline{v} =$$

$$\underline{u}\cdot(\underline{B}\,\underline{C})\,\underline{v} = (\underline{B}\,\underline{C})^T\underline{u}\cdot\underline{v} \qquad \text{CVD}$$

PRODOTTO TENSORIALE

$$(\underline{a}\otimes\underline{b})\,\underline{v} = (\underline{b}\cdot\underline{v})\,\underline{a}$$

1 vett. associa altro vett ($\parallel$ a $\underline{a}$). - Ver. se è Lin:

$$(\underline{a}\otimes\underline{b})(\alpha_1\underline{v}_1+\alpha_2\underline{v}_2) = (\underline{b}\cdot(\alpha_1\underline{v}_1+\alpha_2\underline{v}_2))\,\underline{a} =$$

$$= (\alpha_1(\underline{b}\cdot\underline{v}_1)+\alpha_2(\underline{b}\cdot\underline{v}_2))\,\underline{a} = \alpha_1(\underline{b}\cdot\underline{v}_1)\,\underline{a} +$$

$$\alpha_2(\underline{b}\cdot\underline{v}_2)\,\underline{a} = (\underline{a}\otimes\underline{b})\,\alpha_1\underline{v}_1 + (\underline{a}\otimes\underline{b})\,\alpha_2\underline{v}_2 \qquad \text{CVD}$$

- Coppia vettori collegati da $\otimes$ si chiama DIADE

- Composizione di 2 prod. tens.

$$(\underline{a}\otimes\underline{b})(\underline{c}\otimes\underline{d}) = (\underline{b}\cdot\underline{c})\,\underline{a}\otimes\underline{d}$$

$$(\underline{a}\otimes\underline{b})(\underline{c}\otimes\underline{d})\,\underline{u} = (\underline{a}\otimes\underline{b})(\underline{d}\cdot\underline{u})\,\underline{c} =$$

$$(\underline{d}\cdot\underline{u})(\underline{b}\cdot\underline{c})\,\underline{a} = (\underline{b}\cdot\underline{c})(\underline{a}\otimes\underline{d})\,\underline{u}$$

- $(\underline{a} \otimes \underline{b})^T = \underline{b} \otimes \underline{a}$. Dim:

$$\underline{u} \cdot (\underline{a} \otimes \underline{b}) \underline{v} = \underline{u} \cdot \underline{a} \, (\underline{b} \cdot \underline{v}) = (\underline{b} \otimes \underline{a}) \underline{u} \cdot \underline{v}$$

Deve essere anche $\underline{u} \cdot (\underline{a} \otimes \underline{b}) = (\underline{a} \otimes \underline{b})^T \underline{u} \cdot \underline{v}$ CVD

· Componenti di $(\underline{a} \otimes \underline{b})_{mn}$ → non usare sempre i, j

$$= (\underline{a} \otimes \underline{b}) \underline{e}_n \cdot \underline{e}_m = (\underline{a} \cdot \underline{e}_m)(\underline{b} \cdot \underline{e}_n) = a_m b_n$$

$$\left[(\underline{a} \otimes \underline{b})_{ij}\right] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

$$(\underline{u} \otimes \underline{v})_{23} = u_2 v_3 \; ; \; (\underline{u} \otimes \underline{v})_{kl} = u_k v_l$$

Hp: $\underline{A} = A_{ij} \, \underline{e}_i \otimes \underline{e}_j$ pari a $A_{11} \, \underline{e}_1 \otimes \underline{e}_1 + A_{12} \, \underline{e}_1 \otimes \underline{e}_2 + A_{13} \, \underline{e}_1 \otimes \underline{e}_3 + A_{21} \, \underline{e}_2 \otimes \underline{e}_1 + \dots + A_{33} \, \underline{e}_3 \otimes \underline{e}_3$

Dim:

$(A_{ij} \, \underline{e}_i \otimes \underline{e}_j) \underline{u}$ pari a $\underline{v} = A_{ij} u_j \underline{e}_i$

$v_i = A_{ij} u_j$ CVD

- Tensore identico

$$\underline{I} \, \underline{u} = \underline{u} \; ; \quad [I_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{I}} = \underline{e}_1 \otimes \underline{e}_1 + \underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3$$

$$\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \otimes \underline{e}_j \quad \text{Dim.}$$

$$\underline{\underline{I}}\, \underline{u} = \left(\delta_{ij}\, \underline{e}_i \otimes \underline{e}_j\right) \underline{u} = \delta_{ij}\, \underline{e}_i\, u_j = u_i\, \underline{e}_i = \underline{u} \,!$$

PRODOTTO SCALARE DI TENSORI

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = A_{ij}\, B_{ij}$$

pari a $A_{11} B_{11} + A_{12} B_{12} + \dots + A_{33} B_{33}$

È analogo al p.s. dei vettori:

$$\underline{u} \cdot \underline{v} = u_i v_i = u_1 v_1 + \dots + u_3 v_3$$

$$\cdot \quad \underline{\underline{A}} \cdot \underline{\underline{B}}\, \underline{\underline{C}} = \underline{\underline{B}}^T \underline{\underline{A}} \cdot \underline{\underline{C}}$$

$$A_{ij} \left(B_{ik} C_{kj}\right) = \left(B_{ik} A_{ij}\right) C_{kj} = \left(\underline{\underline{B}}^T \underline{\underline{A}}\right)_{kj} C_{kj} =$$

$$= \underline{\underline{B}}^T \underline{\underline{A}} \cdot \underline{\underline{C}}$$

$$\left[\left(\underline{\underline{B}}^T \underline{\underline{A}}\right)_{kj} = \left(\underline{\underline{B}}^T\right)_{km} A_{mj} = B_{mk} A_{mj}\right]$$

$$\underline{\underline{A}}\, \underline{\underline{B}} \cdot \underline{\underline{C}} = \underline{\underline{B}} \cdot \underline{\underline{A}}^T \underline{\underline{C}}$$

$$= \underline{\underline{A}}\, \underline{\underline{C}}^T \cdot \underline{\underline{B}}$$

[P.S. TENS PER UNA DIADE]

$$\underline{\underline{A}} \cdot \left(\underline{a} \otimes \underline{b}\right) = A_{ij}\, a_i\, b_j = \underline{\underline{A}}\, \underline{b} \cdot \underline{a}$$

$$A_{pq} = \underline{\underline{A}}\, \underline{e}_q \cdot \underline{e}_p = \underline{\underline{A}} \cdot \left(\underline{e}_p \otimes \underline{e}_q\right)$$

$$v_p = \underline{v} \cdot \underline{e}_p \quad \text{(analogo!)} \qquad \text{base di } \mathcal{V}$$

(12) $\underline{\underline{A}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j$

P.S. di 2 DIADI

$$(\underline{a} \otimes \underline{b}) \cdot (\underline{c} \otimes \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$$

P.S. di un sym e un skw

$S \in sym$; $W \in skw$ $\qquad S_{ij} = S_{ji}$

$$\underline{S} \cdot \underline{W} = S_{ij} W_{ij} = 0 \qquad W_{ij} = -W_{ji}$$

I due spazi di S e W sono $\perp$

In $\underline{A}$ abbiamo come base $\underline{e}_i \otimes \underline{e}_j$

In $\underline{S}$ " " " $\frac{1}{\sqrt{2}}(\underline{e}_i \otimes \underline{e}_j + \underline{e}_j \otimes \underline{e}_i)$

In $\underline{W}$ " " " $\frac{1}{\sqrt{2}}(\underline{e}_i \otimes \underline{e}_j - \underline{e}_j \otimes \underline{e}_i)$

Lin ha dimens. 9 $= 6 + 3$

Sym " " 6 $\quad$ Sym $\oplus$ Skw

Skw " " 3

Essendo Skw a dim. 3, usciamo corr. biunivoca con spazio vettori:

$$\underline{W}\,\underline{a} = \underline{w} \times \underline{a}$$

Ex. $\underline{W}\,\underline{e}_1 \cdot \underline{e}_2 = \underline{w} \times \underline{e}_1 \cdot \underline{e}_2$. Posso fare permut. ciclica dei vettori: $= \underline{e}_1 \times \underline{e}_2 \cdot \underline{w} = \underline{e}_3 \cdot \underline{w} = w_3 =$

$= W_{21} = -W_{12}$

$$W_{23} = \underline{W}\,\underline{e}_3 \cdot \underline{e}_2 = \underline{W} \times \underline{e}_3 \cdot \underline{e}_2 = -\underline{e}_1 \cdot \underline{w} = -w_1$$

Quindi $w_1 = W_{32} = -W_{23}$

$w_2 = W_{13} = -W_{31}$ (incrementi indici di 1)

$$[W_{ij}] = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

Quando $\underline{W}\,\underline{a} = \underline{w} \times \underline{a}$, $\forall\, \underline{a} \in V$

$$\underline{A} = \operatorname{sym} \underline{A} + \underbrace{\operatorname{skw} \underline{A}}$$

vettore associato è pari a $\frac{1}{2}(\underline{e}_i \times \underline{A}\,\underline{e}_i)$

Applichiamo $\frac{1}{2}(\underline{e}_i \times \underline{A}\,\underline{e}_i) \times \underline{u} =$

$$\left[(\underline{w} \times \underline{v}) \times \underline{u} = \underline{v}(\underline{w} \cdot \underline{u}) - \underline{w}(\underline{u} \cdot \underline{v})\right]$$

$$= \frac{1}{2}\underline{A}\,\underline{e}_i(\underline{u} \cdot \underline{e}_i) - \frac{1}{2}\underline{e}_i(\underline{A}\,\underline{e}_i \cdot \underline{u}) =$$

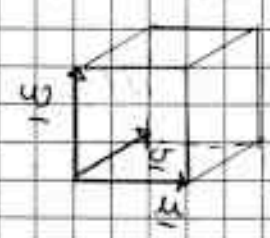
$$= \frac{1}{2}\underline{A}\,(\underbrace{u_i\,\underline{e}_i}_{\underline{u}}) - \frac{1}{2}\underline{e}_i(\underbrace{\underline{A}^T\underline{u}}_{\underline{e}_i \cdot \underline{A}^T\underline{u}})_i =$$

$$= \frac{1}{2}\underline{A}\,\underline{u} - \frac{1}{2}\underline{A}^T\underline{u} = \frac{1}{2}(\underline{A} - \underline{A}^T)\underline{u} = (\operatorname{skw} \underline{A})\underline{u}$$

PRODOTTO MISTO

$$\underline{u} \times \underline{v} \cdot \underline{w} = \varepsilon_{ijk}\, u_i\, v_j\, \underline{e}_k \cdot \underline{w} = \varepsilon_{ijk}\, u_i\, v_j\, w_k =$$

$$= \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$ ovvero il volume di



Nullo se 2 vett // o complanari

det $\underline{A}$

Data terna di vett. lin indip. $\underline{u}, \underline{v}, \underline{w}$ (p. misto $\neq 0$) allora: $[\underline{e}_1, \underline{e}_2, \underline{e}_3]$

$$\det \underline{A} = \frac{\underline{A}\underline{u} \times \underline{A}\underline{v} \cdot \underline{A}\underline{w}}{\underline{u} \times \underline{v} \cdot \underline{w}} =$$

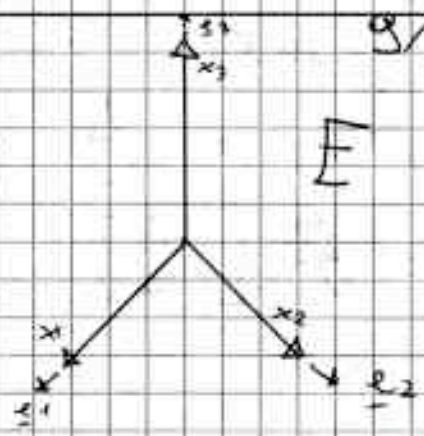
$$= \frac{\underline{A}\underline{e}_1 \times \underline{A}\underline{e}_2 \cdot \underline{A}\underline{e}_3}{[\underline{e}_1 \times \underline{e}_2 \cdot \underline{e}_3] \to 1} = [\underline{A}\underline{e}_1 = A_{k1}\underline{e}_k]$$

$$= \det \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Da la misura della distorsione provocata da A

9/3/09

$\underline{v} \in V$



Base sp. vett. $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$

Compon. in sist. di coord tramite $v_i = \underline{v} \cdot \underline{e}_i$, $i = 1,2,3$

E si può scrivere $\underline{v} = v_i \underline{e}_i =$

$v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$

Tens del 2° ordine ($\in$ Lin).

$\underline{e}_i \otimes \underline{e}_j$, $i,j = 1,2,3$

$(\underline{A})_{ij} = A_{ij} = \underline{A} \cdot (\underline{e}_i \otimes \underline{e}_j)$, $\underline{A} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j =$

$= A_{11}\, \underline{e}_1 \otimes \underline{e}_1 + A_{12}\, \underline{e}_1 \otimes \underline{e}_2 + \dots + A_{33}\, \underline{e}_3 \otimes \underline{e}_3$

Cambia solo la dim dello spazio

$\underline{u} \cdot \underline{v} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$

$\underline{A} \cdot \underline{B} = A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + \dots + A_{33} B_{33}$]

Traccia del tensore: $\mathrm{tr}\, \underline{A} = \underline{A} \cdot \underline{I} = A_{ij} \delta_{ij} =$
$A_{ii} = A_{11} + A_{22} + A_{33}$

Tutti i risq. x Lin si possono ripetere per le trasf. lineari di Lin in Lin (così come Lin è sp. trasf. lin di V in V).

Si denotano come $\mathbb{A} : \underline{A} \in \mathrm{Lin} \mapsto \mathbb{A}[\underline{A}] \in \mathrm{Lin}$
[parentesi [ ] per ind. funz. lineare]

$$\mathbb{A}[\alpha_1 \underline{A}_1 + \alpha_2 \underline{A}_2] = \alpha_1 \mathbb{A}[\underline{A}_1] + \alpha_2 \mathbb{A}[\underline{A}_2]$$

4 TENSORI DEL 4 ORDINE

$$\mathbb{A}_{ijkl} = \mathbb{A}[\underline{e}_k \otimes \underline{e}_l] \cdot (\underline{e}_i \otimes \underline{e}_j)$$

($A_{ij} = \underline{A}\, \underline{e}_j \cdot \underline{e}_i$, stesso modo)

$\underline{B} = \mathbb{A}[\underline{A}]$; in comp. $B_{ij} = \mathbb{A}_{ijkl} A_{kl}$
$i, j = 1, 2, 3$

- Prodotto tensoriale tra elem. di Lin:

$(\underline{A} \otimes \underline{B})\, \underline{C} = \underline{A} (\underline{B} \cdot \underline{C})$ → ad ogni elem. di Lin associa un elem. di Lin

[A tensore B applicato su C]

Sono tensori del 4° ordine

- Trasp. del tens. 4 ordine:

$$\underline{B} \cdot \mathbb{A}[\underline{A}] = \mathbb{A}^T[\underline{B}] \cdot \underline{A}, \quad \forall A, B \in \mathrm{Lin}$$

$(\mathbb{A}^T)_{ijkm} = \mathbb{A}_{klij}$ (si scambiano a coppie indici)

Dim: $(\mathbb{A}^T)_{ijkl} = \mathbb{A}^T(\underline{e}_k \otimes \underline{e}_l) \cdot \underline{e}_i \otimes \underline{e}_j =$

$= \underline{e}_k \otimes \underline{e}_l \cdot \mathbb{A}[\underline{e}_i \otimes \underline{e}_j] = \mathbb{A}_{klij}$

Si dice che $\mathbb{A}$ è sym se $\mathbb{A}^T = \mathbb{A}$, ovvero

$\mathbb{A}_{ijkl} = \mathbb{A}_{klij}$

Per t. 2 ord., se $A \in$ sym allora $A_{ij} = A_{ji}$

Si dim. che $\mathbb{A} = \mathbb{A}_{ijkl} (\underline{e}_i \otimes \underline{e}_j) \otimes (\underline{e}_k \otimes \underline{e}_l)$

Con qualsiasi $\mathcal{L}$ (spazio vett. H) dove è definito il prod. scal.

Allora cons. $f: \mathcal{L} \xrightarrow{lin} \mathbb{R}$ (def. di linearità)

Quindi $f(\alpha \underline{a} + \beta \underline{b}) = \alpha f(\underline{a}) + \beta f(\underline{b})$

$\exists$ il TEOREMA DI RAPPRESENTAZIONE

$f(\underline{a}) = \underline{u}_f \cdot \underline{a}$: valore della fun. si può ottenere tramite p.s. con vettore unico prefissato $\underline{u}_f$ ($\exists$ unico) $\forall \underline{a} \in \mathcal{L}$

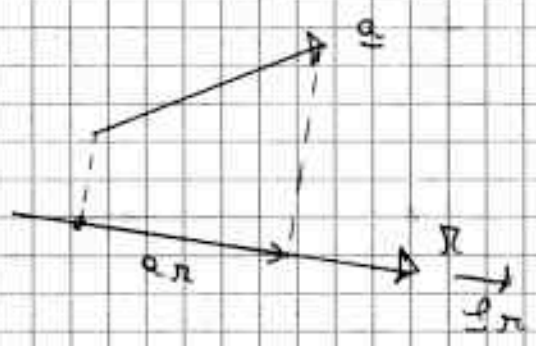
Ex: $\mathcal{V}$

$f(\underline{a}) = a_r$ [componente su retta orient. r]

$\exists$ unico vettore / $f(\underline{a}) =$

$= a_r \Rightarrow \underline{a} \cdot \underline{e}_r = f(\underline{a})$

Qui è proprio il versore di r.



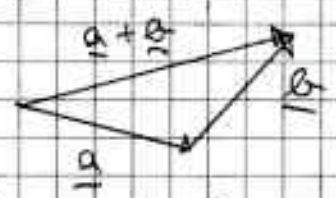
RICHIAMI DI ANALISI

- <u>Norma</u> in spazio lineare $\mathcal{L}$

$$\|\cdot\| : \mathcal{L} \longrightarrow \mathbb{R}_0^+ \quad \text{(reale non negativo)}$$

Proprietà:

- $\|\underline{a}\| \geq 0$, $\|\underline{a}\| = 0 \iff \underline{a} = \underline{0}$
- $\|\lambda \underline{a}\| = |\lambda| \, \|\underline{a}\|$
- $\|\underline{a} + \underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|$ (disug. triangolare)



Noi lavoriamo su spazi E dove è def il p.s.
Norma è indotta dal p.s.

$$\|\underline{a}\| = (\underline{a} \cdot \underline{a})^{1/2} \quad \text{oppure} \quad \|\underline{A}\| = (\underline{A} \cdot \underline{A})^{1/2}$$

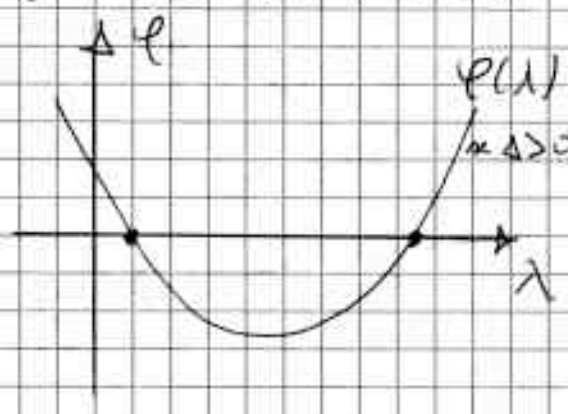
Cons: $(\lambda \underline{a} + \underline{b}) \cdot (\lambda \underline{a} + \underline{b}) \geq 0$

Se $\underline{a}$, $\underline{b}$ assegnati si può dire che è una
$\varphi(\lambda) = (\lambda \underline{a} + \underline{b}) \cdot (\lambda \underline{a} + \underline{b})$ [f. $\mathbb{R}$ a var $\mathbb{R}$]

Sviluppo: $\varphi(\lambda) = \lambda^2 \underline{a} \cdot \underline{a} + 2\lambda \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b} \geq 0$

$$\Delta = (\underline{a} \cdot \underline{b})^2 - (\underline{a} \cdot \underline{a})(\underline{b} \cdot \underline{b}) = (\underline{a} \cdot \underline{b})^2 - \|\underline{a}\|^2 \|\underline{b}\|^2$$

Non può mai essere $\Delta > 0$, altrimenti $\varphi(\lambda)$ ha 2 radici $\mathbb{R} \Rightarrow \varphi(\lambda)$ avrebbe anche valori negativi!



(18) $(\underline{a} \cdot \underline{b})^2 \leq \|\underline{a}\|^2 \, \|\underline{b}\|^2$

$|\underline{a} \cdot \underline{b}| \leq \|\underline{a}\| \, \|\underline{b}\|$

Ora calcoliamo $\|\underline{a}+\underline{b}\|^2 = (\underline{a}+\underline{b}) \cdot (\underline{a}+\underline{b}) =$

$= \underline{a} \cdot \underline{a} + 2\, \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b} = \|a\|^2 + 2a \cdot b + \|\underline{b}\|^2 \leq$

$\leq \|a\|^2 + 2|\underline{a} \cdot \underline{b}| + \|\underline{b}\|^2$. Sostituendo la prima $\leq$

$\|a^2\| + 2\|\underline{a}\|\|\underline{b}\| + \|\underline{b}\|^2 = (\|\underline{a}\| + \|\underline{b}\|)^2$. Allora

$\|\underline{a} + \underline{b}\|^2 \leq (\|\underline{a}\| + \|\underline{b}\|)^2$

$\|\underline{a} + \underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|$

Ci interessa $\mathbb{R}$, E, V Lin, tensori scal e ord.
tutti dotati di p. scal e quindi normabili.

Cons $\varphi : A \to B$

$$\varphi : \underline{x} \in A \longmapsto \varphi(\underline{x}) \in B$$

Cons: Se $\lim_{\underline{x} \to \underline{0}} \dfrac{\|\varphi(\underline{x})\|}{\|\underline{x}\|} = 0$ allora $\varphi(\underline{x})$ è un $\frac{1}{\infty}$ di ordine sup. risp. a x per $\underline{x} \to \underline{0}$

Ovvero $\varphi(\underline{x}) = o(\underline{x})$

$\varphi : A \to B$ è <u>differenziabile</u> in $\underline{x}$

se $\exists$ trasf. lin $L(\underline{x})$ / $\varphi(\underline{y}) - \varphi(\underline{x}) = L(\underline{x})[\underline{y} - \underline{x}] + \underline{\rho}(\underline{y} - \underline{x})$ con $\underline{\rho}(\underline{y} - \underline{x}) = o(\underline{y} - \underline{x})$

la $L(\underline{x})$ è la <u>DERIVATA</u> $D\varphi(\underline{x})$

[Sappiamo che $f(x_0 + \alpha x) - f(x_0) = f'(x_0)\alpha x + o(\alpha x)$ infatti ...]

Deriv. per funz. tra spazi normati nella defin. abbiamo usato concetto che è "proprietà" nelle $f(\mathbb{R}) \mapsto \mathbb{R}$.

Quindi:

$$\lim_{(\underline{y}-\underline{x})\to \underline{0}} \frac{\|\underline{\rho}(\underline{y}-\underline{x})\|}{\|\underline{y}-\underline{x}\|} = 0$$

Vogliamo calcolare la derivata.

Poniamo $\underline{y} - \underline{x} = \lambda \underline{u}$. Supponiamo $\exists$ la deriv.:

$$\varphi(\underline{x} + \lambda \underline{u}) = \varphi(\underline{x}) = D\varphi(\underline{x})[\lambda \underline{u}] + \underline{\rho}(\lambda \underline{u})$$

$$= \lambda D\varphi(x)[\underline{u}]$$

Divid. per λ:

$$\frac{\varphi(\underline{x} + \lambda \underline{u}) - \varphi(\underline{x})}{\lambda} = D\varphi(\underline{x})[\underline{u}] + \frac{\rho(\lambda \underline{u})}{\lambda}$$

Applichiamo il limite di $\lambda \to 0$:

$$D\varphi(\underline{x})[\underline{u}] = \lim_{\lambda\to 0} \frac{\varphi(\underline{x} + \lambda \underline{u}) - \varphi(\underline{x})}{\lambda} - \lim_{\lambda \to 0} \frac{\rho(\lambda \underline{u})}{\lambda}$$

$$= \frac{d}{d\lambda} \varphi(\underline{x} + \lambda \underline{u}) \Big|_{\lambda = 0} - \lim_{\lambda\to 0} \frac{\underline{\rho}(\lambda \underline{u})}{\lambda}$$

Vogliamo 2° term. nullo. Ip: $\lim_{\lambda \to 0} \frac{\underline{\rho}(\lambda \underline{u})}{\lambda} = 0$ con $\rho(\lambda \underline{u}) = o(\lambda \underline{u})$

$$\lim_{\lambda\to 0} \frac{\|\underline{\rho}(\lambda \underline{u})\|}{\|\lambda \underline{u}\|} = \lim_{\lambda \to 0} \frac{\|\underline{\rho}(\lambda \underline{u})\|}{|\lambda| \, \|\underline{u}\|} = \frac{1}{\|\underline{u}\|} \lim_{\lambda\to 0} \frac{\|\underline{\rho}(\lambda \underline{u})\|}{|\lambda|} =$$

$$= \frac{1}{\|\underline{u}\|} \lim_{\lambda \to 0} \left\| \frac{\underline{\rho}(\lambda \underline{u})}{\lambda} \right\|$$

$$\lim_{\lambda \underline{u} \to \underline{0}} \frac{\| \underline{\rho}(\lambda \underline{u}) \|}{\| \lambda \underline{u} \|} = 0 \quad \text{CVD.}$$

Se $\lambda \to 0$, $\lambda \underline{u} \to \underline{0}$!

Infatti $\forall \delta > 0$, $\exists h > 0 /$

$|\lambda| < h \Rightarrow \| \lambda \underline{u} \| < \delta$

(λ piccolo)

Quindi $\forall \varepsilon > 0$, $\exists \delta > 0 / \| \lambda \underline{u} \| < \delta \Rightarrow \dfrac{\| \underline{\rho}(\lambda \underline{u}) \|}{\| \lambda \underline{u} \|} < \varepsilon$

Se il lim. della norma è 0, il vett. è nullo.

$$\boxed{D\varphi(\underline{x})[\underline{u}] = \frac{d}{d\lambda} \varphi(\underline{x} + \lambda \underline{u}) \Big|_{\lambda = 0}}$$

derivata direzionale

Proprietà:

- $D(\varphi \psi)(\underline{x})[\underline{u}] = \dfrac{d}{d\lambda} \left(\varphi(\underline{x} + \lambda \underline{u}) \psi(\underline{x} + \lambda \underline{u}) \right)_{\lambda = 0} =$

$= \left(\dfrac{d}{d\lambda} \varphi(\underline{x} + \lambda \underline{u}) \right)_{\lambda = 0} \psi(\underline{x}) + \varphi(\underline{x}) + \dfrac{d}{d\lambda} \left(\psi(\underline{x} + \lambda \underline{u}) \right)_{\lambda = 0}$

$= D\varphi(\underline{x})[\underline{u}] \psi(\underline{x}) + \varphi(\underline{x}) D\psi(\underline{x})[\underline{u}]$

(anche $\mathbb{R}$ infatti è sp. normato e noi conosc. le regole)

- F. composta: $\chi = \varphi \circ \psi$

$$D\chi(\underline{x})[\underline{u}] = D\varphi(\psi(\underline{x}))[D\psi(\underline{x})[\underline{u}]]$$

Conv. fun. reale del punto.

$\varphi : \mathcal{E} \to \mathbb{R}$ allora si chiama la derivata il GRADIENTE: $D\varphi(x) = \nabla \varphi(x)$

$$\varphi(y) - \varphi(x) = \nabla \varphi(x)[y - x] - \rho(y - x)$$

vettore

Fun. di vettore a valore $\mathbb{R}$, lineare.

$\nabla\varphi : V \xrightarrow{lin} \mathbb{R}$

Secondo il th di rapp. $\exists$ unico vettore $\underline{v}$ /

$\nabla\varphi[\underline{u}] = \underline{v}\cdot\underline{u}$. Questo $\underline{v}$ lo identifichiamo con la derivata, cioè $\nabla\varphi[\underline{u}] = \nabla\varphi\cdot\underline{u}$

Il gradiente di una f. scalare è quindi un vettore

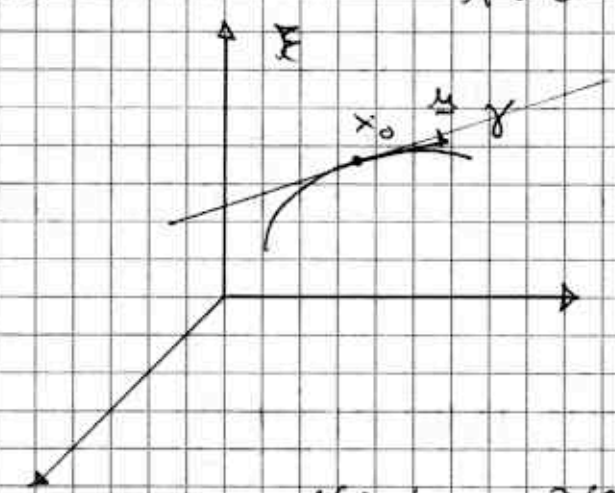
$$\boxed{(\nabla\varphi)_i} = \nabla\varphi\cdot\underline{e}_i = \nabla\varphi[\underline{e}_i] = \frac{d}{d\lambda}\varphi(\underline{x}+\lambda\underline{e}_i)\Big|_{\lambda=0} =$$

$$= \frac{\partial\varphi}{\partial x_h}\frac{dx_h}{d\lambda} = \boxed{\frac{\partial\varphi}{\partial x_i} = \varphi_{,i}}$$ [der. parziale!]

Ad es. $i=1$: $\frac{d}{d\lambda}\varphi(x_1+\lambda, x_2, x_3)\Big|_{\lambda=0} = \frac{\partial\varphi}{\partial x_1}$

$$\nabla\varphi = (\nabla\varphi)_i\,\underline{e}_i = \varphi_{,i}\,\underline{e}_i = \varphi_{,1}\,\underline{e}_1 + \varphi_{,2}\,\underline{e}_2 + \varphi_{,3}\,\underline{e}_3$$

$$\frac{d}{d\lambda}\varphi(\underline{x}+\lambda\underline{u})\Big|_{\lambda=0}$$



Curva $x = x(\lambda)$ /

in $x(0) = x_0$,

$\frac{dx}{d\lambda}(0) = \underline{u}$

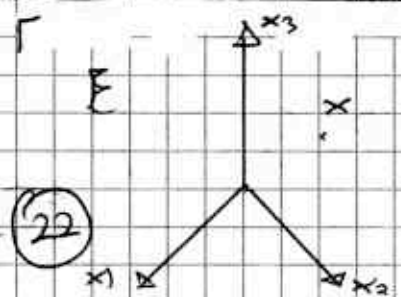
Curva opportunamente parametrizzata.

Si ha $x = x_0 + \lambda\underline{u}$

Facciamo $\frac{d\varphi}{d\lambda}\Big|_{\lambda=0} = \frac{\partial\varphi}{\partial x_h}\frac{dx_h}{d\lambda} = \frac{\partial\varphi(x_0)}{\partial x_h}u_h = \nabla\varphi(x_0)[\underline{u}]$

$\hookrightarrow \nabla\varphi(x_0)\cdot\underline{u}$

10/3/09



$\varphi : E \to \mathbb{R}$

$\nabla\varphi = \varphi_{,i}\,\underline{e}_i$

$\underline{v} : E \to V$

$$\underset{\text{vett.}}{\underline{v}(y)} - \underline{v}(x) = \underset{\text{trasf. lineare}}{\nabla \underline{v}(x)}\underset{\text{vett.}}{[y-x]} + \underline{f}(y-x)$$

$$(\nabla \underline{v})_{ij} = \nabla \underline{v} \cdot (\underline{e}_i \otimes \underline{e}_j) = \underbrace{\nabla \underline{v}[\underline{e}_j]} \cdot \underline{e}_i =$$

$$= (v_k \underline{e}_k)_{,j} \cdot \underline{e}_i = \qquad \left[\nabla \underline{v}(x)[\underline{e}_j] = \frac{d}{d\lambda} \underline{v}(x + \lambda \underline{e}_j)\Big|_{\lambda=0} \right.$$

$$= v_{k,j} \cdot \delta_{ki} = v_{i,j} \qquad \text{ovvero } \frac{\partial \underline{v}}{\partial x_j} = \underline{v}_{,j} \qquad \Big]$$

$$[\nabla \underline{v}] = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix}, \quad (\nabla \underline{v})_{ij} = v_{i,j}$$

allora

$$\nabla \underline{v} = (\nabla \underline{v})_{ij}\, \underline{e}_i \otimes \underline{e}_j = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j =$$

$$= (v_i \underline{e}_i)_{,j} \otimes \underline{e}_j = \underline{v}_{,i} \otimes \underline{e}_i = \underline{v}_{,1} \otimes \underline{e}_1 +$$

$$+ \underline{v}_{,2} \otimes \underline{e}_2 + \underline{v}_{,3} \otimes \underline{e}_3$$

Quindi grad. f. scalare è $\nabla \varphi = \varphi_{,i}\, \underline{e}_i$

" " " vett. è $\nabla \underline{v} = \underline{v}_{,i} \otimes \underline{e}_i$

- $\nabla(\underline{u} \cdot \underline{v}) = (\underline{u} \cdot \underline{v})_{,k}\, \underline{e}_k = (\underline{u}_{,k} \cdot \underline{v} +$

$\underline{u} \cdot \underline{v}_{,k})\, \underline{e}_k = (\underline{u}_{,k} \cdot \underline{v})\, \underline{e}_k + (\underline{u} \cdot \underline{v}_{,k})\, \underline{e}_k =$

(rapp. tensore come diade) $= (\underline{e}_k \otimes \underline{u}_{,k})\, \underline{v} +$

$+ (\underline{e}_k \otimes \underline{v}_{,k})\, \underline{u} = [$ sappiamo che $(\underline{a} \otimes \underline{b})^T = \underline{b} \otimes \underline{a}]$

$= \nabla \underline{u}^T [\underline{v}] + \nabla \underline{v}^T [\underline{u}]$.

- $\nabla(\varphi \underline{v}) = (\varphi \underline{v})_{,n} \otimes \underline{e}_n = (\varphi_{,n} \underline{v} + \varphi \underline{v}_{,n}) \otimes \underline{e}_n =$

$= \varphi_{,n} \underline{v} \otimes \underline{e}_n + \varphi (\underline{v}_{,n} \otimes \underline{e}_n) = \underline{v} \otimes \nabla\varphi + \varphi \nabla \underline{v}$

$[\operatorname{tr} \underline{A} = \underline{A} \cdot \underline{I} = A_{mn} \delta_{mn} = A_{mm} = A_{11} + A_{22} + A_{33}]$

- DIVERGENZA DI UN VETTORE $[\underline{v}]$

$\underline{\operatorname{div} \underline{v} = \operatorname{tr} \nabla \underline{v}^*} = \nabla \underline{v} \cdot \underline{I} = (\nabla \underline{v})_{ij} \delta_{ij} =$

$= v_{i,j} \delta_{ij} = v_{i,i} = v_{1,1} + v_{2,2} + v_{3,3}$

- DIVERGENZA DI UN TENSORE $[\underline{A}]$

$\underline{\operatorname{div} \underline{A} \cdot \underline{a} = \operatorname{div} (\underline{A}^T \underline{a})} \quad \forall \underline{a} \in V$

vettore

Si può anche esprimere $\operatorname{div} \underline{v} = \nabla \underline{v} \cdot \underline{I} = (\underline{v}_{,n} \otimes \underline{e}_i) \cdot \underline{I}$

$= \underline{I}\, \underline{e}_i \cdot \underline{v}_{,i} = \underline{v}_{,i} \cdot \underline{e}_i = \underline{v}_{,1} \cdot \underline{e}_1 + \underline{v}_{,2} \cdot \underline{e}_2 + \underline{v}_{,3} \cdot \underline{e}_3 \Rightarrow$

$\underline{\operatorname{div} \underline{v} = \underline{v}_{,m} \cdot \underline{e}_m}$

Allora: [con $\underline{a}$ = costante]

$\operatorname{div} \underline{A} \cdot \underline{a} = \operatorname{div} (\underline{A}^T \underline{a}) = (\underline{A}^T \underline{a})_{,i} \cdot \underline{e}_i = \underline{A}^T_{,i} \underline{a} \cdot \underline{e}_i =$

$= \boxed{\underline{a} \cdot \underline{A}_{,i} \cdot \underline{e}_i = \underline{a} \cdot \operatorname{div} \underline{A}}$

$\underline{\operatorname{div} \underline{A} = \underline{A}_{,m} \underline{e}_m}$

- $(\operatorname{div} \underline{A})_n = \operatorname{div} \underline{A} \cdot \underline{e}_n = (\underline{A}_{,i} \underline{e}_i) \cdot \underline{e}_n =$

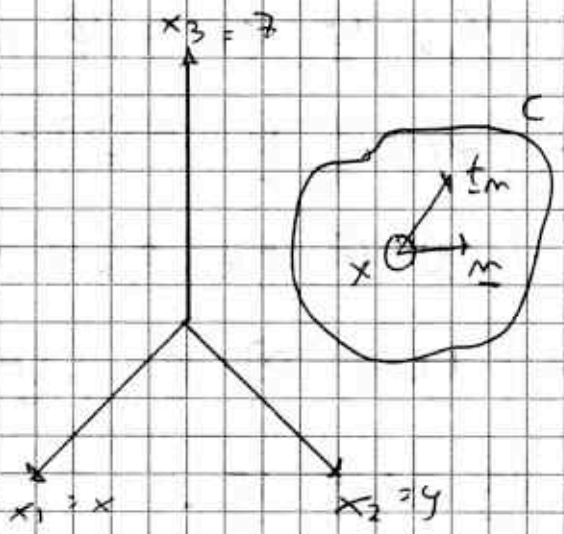
(24) $= \underline{A}_{,i} \cdot (\underline{e}_n \otimes \underline{e}_i) = (A_{mn,i}\, \underline{e}_m \otimes \underline{e}_n) \cdot$

• $\underline{e}_k \otimes \underline{e}_i = \left[\text{ricorda: } (\underline{a} \otimes \underline{b}) \cdot (\underline{c} \otimes \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) \right]$

$= A_{mm,i} \cdot \delta_{mk} \cdot \delta_{mi} = A_{ki,i} = A_{k1,1} + A_{k2,2} +$

$+ A_{k3,3}$

Ex: $\underline{T}(x)\,\underline{n} = \underline{t}_{\underline{n}}(x)$



Di solito usiamo

$$\begin{bmatrix} t_{nx} \\ t_{ny} \\ t_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$\downarrow$

Matrice del tensore degli sforzi

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

Eq. di equilibrio corpo cont.:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x = 0$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0$$

Esprimibile come $\operatorname{div} \underline{T} + \underline{b} = \underline{0}$:

$T_{11,1} + T_{12,2} + T_{13,3} + b_1 = 0$

$T_{21,1} + T_{22,2} + T_{23,3} + b_2 = 0$

$T_{31,1} + T_{32,2} + T_{33,3} + b_3 = 0$

Not. indiciale:

$T_{is,s} + b_i = 0$ per $i = 1,2,3$

* $\operatorname{div}(\varphi \underline{v}) = (\varphi \underline{v})_{,k} \cdot \underline{e}_k = (\varphi_{,k} \underline{v} + \varphi \underline{v}_{,k}) \cdot \underline{e}_k$

$= \varphi_{,k} \underline{v} \cdot \underline{e}_k + \varphi \underline{v}_{,k} \cdot \underline{e}_k = \underline{v} \cdot \nabla\varphi + \varphi \operatorname{div} \underline{v}$.

* $\operatorname{div}(\underline{u} \otimes \underline{v}) = (\underline{u} \otimes \underline{v})_{,k} \underline{e}_k = (\underline{u}_{,k} \otimes \underline{v} +$

$+ \underline{u} \otimes \underline{v}_{,k}) \underline{e}_k = \underline{u}_{,k} (\underline{v} \cdot \underline{e}_k) + \underline{u} (\underline{v}_{,k} \cdot \underline{e}_k) =$

$= (\underline{u}_{,k} \otimes \underline{e}_k) \underline{v} + (\underline{v}_{,k} \cdot \underline{e}_k) \underline{u} = \nabla\underline{u}[\underline{v}] +$

$\underline{u} \operatorname{div} \underline{v}$.

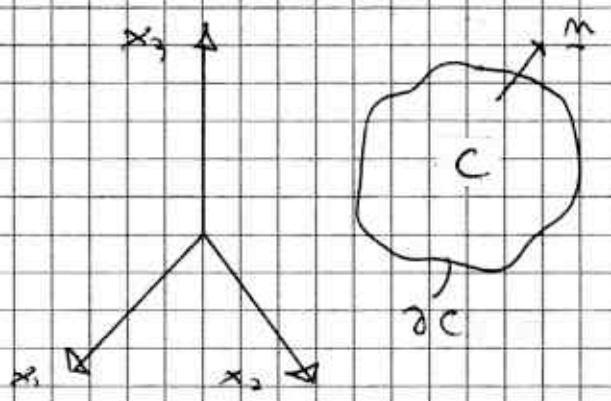
* $\operatorname{div}(\varphi \underline{A}) = (\varphi \underline{A})_{,k} \underline{e}_k = (\varphi_{,k} \underline{A} + \varphi \underline{A}_{,k}) \underline{e}_k =$

$= \varphi_{,k} \underline{A} \underline{e}_k + \varphi \underline{A}_{,k} \underline{e}_k = \underline{A} \nabla\varphi + \varphi \operatorname{div} \underline{A}$

TEOREMA DELLA DIVERGENZA

- <u>Vettori</u>:

[frontiera di corpo regolare]



$$\boxed{\int_C \operatorname{div} \underline{v} \, dV = \int_{\partial C} \underline{v} \cdot \underline{n} \, dS}$$

$\int_C (v_{1,1} + v_{2,2} + v_{3,3}) \, dV =$

$\int_{\partial C} (v_1 n_1 + v_2 n_2 + v_3 n_3) \, dS$

(26) Cons. $v_1 = f$, $v_2 = v_3 = 0$.

$$\int_C \frac{\partial f}{\partial x_1}\, dV = \int_{\partial C} f n_1 \, dS \quad \text{[formula di Gauss - Green]}$$

- Tensori:

$$\int_C \operatorname{div} \underline{A}\, dV = \int_{\partial C} \underline{A}\, \underline{n} \cdot dS$$

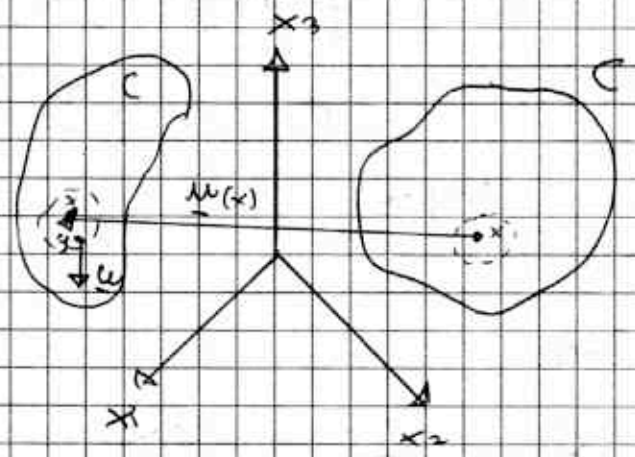
$$\underline{a} \cdot \int_C \operatorname{div} \underline{A}\, dV = \int_C \operatorname{div} \underline{A} \cdot \underline{a} = \int_C \operatorname{div} (\underline{A}^T \underline{a}) =$$

$$= \int_{\partial C} \underline{A}^T \underline{a} \cdot \underline{n}\, dS = \int_{\partial C} \underline{a} \cdot \underline{A}\, \underline{n}\, dS = \underline{a} \cdot \int_{\partial C} \underline{A}\, \underline{n}\, dS$$

$$\int_C A_{ij,j}\, \underline{e}_i\, dV = \int_{\partial C} A_{ij}\, n_j\, \underline{e}_i\, dS$$

$$\int_C (A_{i1,1} + A_{i2,2} + A_{i3,3})\, dV = \int_{\partial C} (A_{i1} n_1 + A_{i2} n_2 + A_{i3} n_3)\, dS$$

RICHIAMI DI TEORIA DELL' ELASTICITÀ



Corpo 3D che subisce deformazione.

Vogliamo spost. $\underline{u}(y)$

[y è nell' intorno di x]

$$\underline{u}(y) - \underline{u}(x) = \nabla \underline{u}(x)[y - x] + \underline{\rho}(y - x)$$

↳ è elem. di Lin, lo decompongo

$\underline{A} = \operatorname{sym} \underline{A} + \operatorname{skw} \underline{A}$. Pongo:

$\underline{E} = \operatorname{sym} \nabla \underline{u} = \frac{1}{2} (\nabla \underline{u} + \nabla \underline{u}^T)$

$$\underline{W} = \operatorname{skw} \nabla \underline{u} = \frac{1}{2}\left(\nabla \underline{u} - \nabla \underline{u}^T\right)$$

Allora:

$$u(\underline{y}) = \underline{u}(x) + \underline{W}(x)[y - x] + \underline{E}(x)[y - x]$$

↳ ad ogni skw associo un vettore

$$u(\underline{y}) = \underbrace{\underline{u}(x) + \underline{\omega}(x) \times (y - x)}_{\text{spostamento rigido}} + \underline{E}(x)[y - x]$$

TENSORE DI DEFORMAZIONE $\frac{1}{\infty}$

$$[E_{ij}] = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) & \frac{1}{2}(u_{1,3} + u_{3,1}) \\ & u_{2,2} & \frac{1}{2}(u_{2,3} + u_{3,2}) \\ \text{sym} & & u_{3,3} \end{bmatrix} =$$

$$\left[E_{ij} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) \cdot \underline{e}_i \otimes \underline{e}_j = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \right]$$

$$\left[(u, v, w) = (u_1, u_2, u_3); \quad (x, y, z) = (x_1, x_2, x_3) \right]$$

$$= \begin{bmatrix} \varepsilon_1 & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ & \varepsilon_2 & \frac{1}{2}\gamma_{23} \\ \text{sym} & & \varepsilon_3 \end{bmatrix}$$

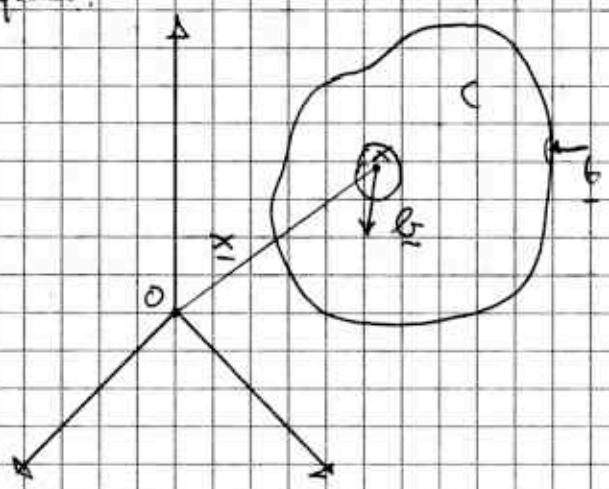
Con:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x_1}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2E_{12}$$

È il tensore che descrive la deformazione

Eq. equil.:

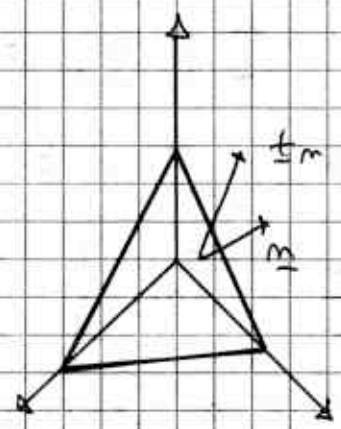


$$\int_C \underline{b}\, dV + \int_{\partial C} \underline{t}\, dS = \underline{0}$$

$$\int_C \underline{x} \times \underline{b}\, dV + \int_{\partial C} \underline{x} \times \underline{t}\, dS = \underline{0}$$

[eq. rid. ed eq. momenti]

La si applica al tetraedro:
si ha il TETRAEDRO DI CAUCHY



$$\boxed{\underline{t}_n = \underline{T}\, \underline{n}}$$

$$\int_C \underline{b}\, dV + \int_{\partial C} \underline{T}\, \underline{n}\, dS = 0$$

↳ applico Th. divergenza:

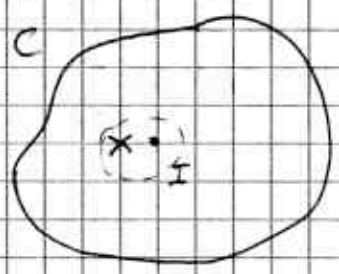
$$\int_C \underline{b}\, dV + \int_C \operatorname{div} \underline{T}\, dV = \underline{0} \Rightarrow \int_C \overbrace{(\operatorname{div} \underline{T} + \underline{b})}^{\underline{a}}\, dV = \underline{0}$$

vera $\forall$ parte di C

$$\Downarrow$$

$$\boxed{\operatorname{div} \underline{T} + \underline{b} = \underline{0}} \text{ in } C$$

$$\int_C \underline{a}\, dV = \underline{0} \quad \text{ovvero} \quad \underline{e}_i \int_C a_i\, dV = \underline{0}$$



$a_1 \neq 0$ Cons. interno di x

$a_1 > 0$

$$\int_I a_1\, dV > 0 \quad \text{No!}$$

Non può accadere (sempre se $\underline{a}$ cont.)

$$\int_C \underline{x} \times \underline{b}\, dV + \int_{\partial C} \underline{x} \times \underline{T}\, \underline{n}\, dS = 0$$

$$\int_{\partial C} \underline{x} \times \underline{T}\,\underline{n} = \left[\int_{\partial C} \underline{x}\,\underline{T}\,\underline{n} = \int_{\partial C} (\underline{x}\,\underline{T})\,\underline{n} = \int_C \operatorname{div}(\underline{x}\,\underline{T}) =\right.$$

$$= \int_C (\underline{x}\,\underline{T})_{,i} \cdot \underline{e}_i = \int_C (\underline{x}\,\underline{T}\,\underline{e}_i)_{,i} = \left.\int_C (\underline{x} \times \underline{T}\,\underline{e}_i)_{,i} =\right]$$

$$= \int_C (\underline{x} \times \underline{T})_{,i} \cdot \underline{e}_i$$

$$(\underline{x} \times \underline{T})_{,i}\,\underline{e}_i = \underline{x}_{,i} \times \underline{T}\,\underline{e}_i + \underline{x} \times \underline{T}_{,i}\,\underline{e}_i =$$

$$= \left[\underline{x}_{m,i} \cdot \underline{e}_m = \underline{e}_i \text{ essendo } \underline{x}_{,i} = (x_m\,\underline{e}_m)_{,i} \text{ e } \frac{\partial x_m}{\partial x_i} = \delta_{mi}\right] = \underline{e}_i \times \underline{T}\,\underline{e}_i + \underline{x} \times \operatorname{div}\underline{T}$$

$$\int_C \underline{x} \times \underline{b}\,dV + \int_C (\underline{e}_i \times \underline{T}\,\underline{e}_i + \underline{x} \times \operatorname{div}\underline{T})\,dV = \underline{0}$$

$$\int_C \left[\underline{x} \times (\underbrace{\operatorname{div}\underline{T}}_{0} + \underline{b}) + \underline{e}_i \times \underline{T}\,\underline{e}_i\right] dV = \underline{0}$$

$$\int_C \underline{e}_i \times \underline{T}\,\underline{e}_i = \underline{0} \quad \text{poiché vale } \forall\, n.$$

$$\Downarrow$$

$$\underline{e}_i \times \underline{T}\,\underline{e}_i = \underline{0} \quad \text{e quindi anche } \frac{1}{2}\,\underline{e}_i \times \underline{T}\,\underline{e}_i = \underline{0}$$

ovvero, essendo il vett. associato a skw $\underline{T}$

$$\operatorname{skw}\underline{T} = \underline{0} \Rightarrow \boxed{\underline{T} = \underline{T}^T} \qquad T_{ij} = T_{ji}$$

è SIMMETRICO

(dovuta alla nullità del momento risultante delle forze agenti sul corpo).

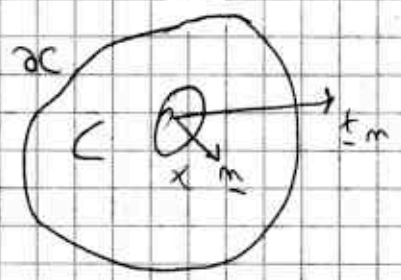
16/03/09

$\underline{\varepsilon} = \text{sym}\, \nabla \underline{u} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$

$(\nabla \underline{u})_{ij} = u_{i,j}$

$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$; ex. $\varepsilon_{11} = u_{1,1} = \varepsilon_1$ (dilat.)

$\varepsilon_{12} = \frac{1}{2}\gamma_{12}$ (scorrimento)



$\underline{T}(x)\,\underline{n} = \underline{t}(x, \underline{n})$ (vett. x unità di sup.)

Eq. equil.:

$$\begin{cases} \text{div}\, \underline{T} + \underline{b} = \underline{0} \\ \underline{T}^T = \underline{T} \end{cases} \quad \text{in } C$$

Cond. al contorno $\partial C = \partial C_1 \cup \partial C_2$ e $\partial C_1 \cap \partial C_2 = 0$

Spesso $\underline{u} = \hat{\underline{u}}$ su ∂C_1 (spost. impr.)

$\underline{T}\,\underline{n} = \hat{\underline{t}}$ su ∂C_2 (assegno tens.)

Legame tra sforzo e deformazione espresso da trasf. lineare, un tensore del 4 ordine: $\underline{T} = \mathbb{C}[\underline{\varepsilon}] = \mathbb{C}[\text{sym}\, \nabla \underline{u}]$

↳ TENSORE DI ELASTICITÀ

Sappiamo che $\mathbb{C}[\underline{W}] = \underline{0}$, $\forall\, \underline{W} \in \text{Skew}$ (a spost. rigido non si associano deformazioni)

Quindi $\mathbb{C}[\underline{\varepsilon}] = \mathbb{C}[\nabla \underline{u}]$ (posso essendo operat. lineare ridurre $\underline{\varepsilon}$ nella parte sym e skew e scrivere).

$T_{ij} = \mathbb{C}_{ijkl}\, \varepsilon_{kl}$; anche $\mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}$

$C_{ijkl} = C_{jikl}$

Si assume $\mathbb{C} = \mathbb{C}^T$. Quindi $\mathbb{C}[\underline{A}] \cdot \underline{B} = \underline{A} \cdot \mathbb{C}[\underline{B}]$, $\forall\, \underline{A}, \underline{B} \in Lin$

In comp. (detta "elasticità") → o anche MODULI ELASTICI

$C_{ijkl} = C_{klij}$, essendo $C_{ijkl} = \mathbb{C}[\underline{e}_k \otimes \underline{e}_l] \cdot (\underline{e}_i \otimes \underline{e}_j) =$

$= (\underline{e}_k \otimes \underline{e}_l) \cdot \mathbb{C}[\underline{e}_i \otimes \underline{e}_j] = C_{klij}$

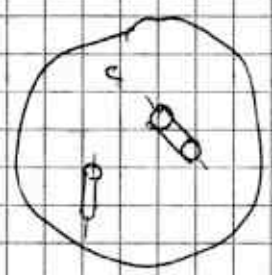
$\mathbb{C}$ avrebbe 81 componenti (3^4). Ma le sym ↓ n. comp. distinte.

Si può anche dire $\mathbb{C}: \underset{\text{sim. 6}}{Sym} \longmapsto \underset{\text{sim. 6}}{Sym}$ e quindi la relativa trasf. ha $6^2 = 36$ comp.

Inoltre $\mathbb{C}$ è sym. Quindi $\frac{(36-6)}{2} + 6 = 21$

Al max 21 comp. distinte di $\mathbb{C}$.

Si possono ridurre ancora ∃ le sym dei materiali.



Ex: 2 prove su 2 parti con dir. <> e stesso comp., ho materiale ISOTROPO.

Se moduli elast. ≠ dal punto corpo è OMOGENEO.

$\underline{T} = \mathbb{C}[\underline{E}]$

$$\left.\begin{aligned}
\sigma_x &= 2G\,\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z);\\
\sigma_y &= 2G\,\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z);\\
\sigma_z &= 2G\,\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z);\\
\tau_{xy} &= G\gamma_{xy};\ \tau_{yz} = G\gamma_{yz};\ \tau_{zx} = G\gamma_{zx}
\end{aligned}\right\}\ \text{Eq. di Lamé}$$

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In maniera equiv. abbiamo:

$$\underline{\underline{T}} = 2\mu \underline{\underline{\varepsilon}} + \lambda (\mathrm{tr}\, \underline{\underline{\varepsilon}}) \underline{\underline{I}}$$

$$T_{ij} = 2\mu\, \varepsilon_{ij} + \lambda (\varepsilon_{kk})\, \delta_{ij}$$

μ, λ: MODULI DI LAMÉ

$\rightarrow \mu = G$

MODULO DI ELASTICITÀ TRASVERSALE

Ex: $T_{11} = 2\mu\, \varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$

$T_{12} = 2\mu\, \varepsilon_{12}$

$$\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right) \\
\varepsilon_y &= \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right) \\
\varepsilon_z &= \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right) \\
\gamma_{xy} &= \frac{1}{G}\tau_{xy}\,; \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}\,; \quad \gamma_{zx} = \frac{1}{G}\tau_{zx}
\end{aligned}\right\} \text{Eq. di NAVIER}$$

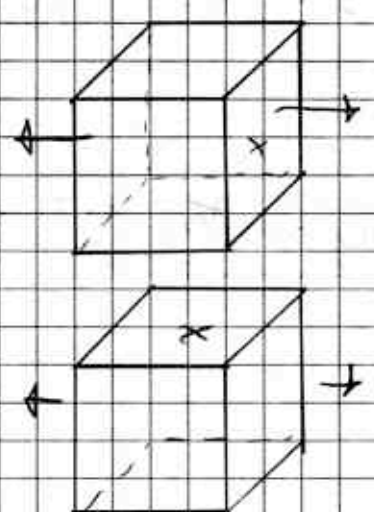
MODULO DI YOUNG

Attento:

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right); \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

quindi ci vuole il 2.

Ma si usano anche materiali non isotropi.

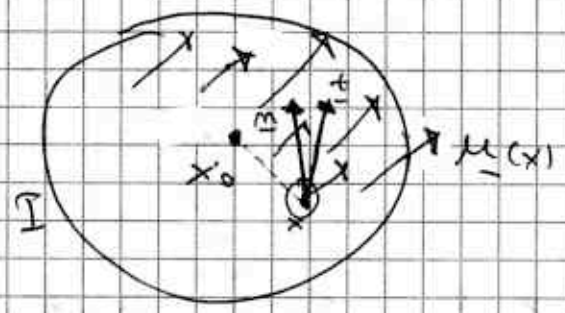


Ex: provino a trat. su faccia x. Poi però ruotiamo e ripetiamo prova e ho stessi risultati: lo trasf. di sym materiale

$\downarrow$

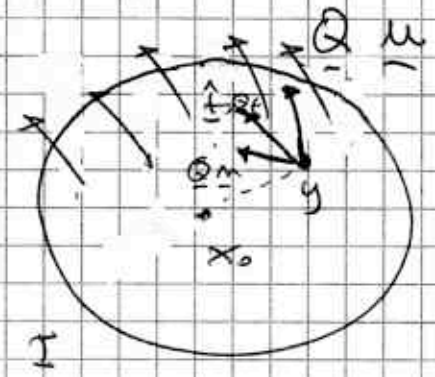
materiale ha sym.

Espress. matematica!



Con. interno I a punto x_0 è corpo.
Applichiamo campo di spostamento. Ho:
$\underline{E}(x)$; $\underline{T}(x)$; $\underline{t}(x, \underline{n}) = \underline{T}(x)\,\underline{n}$

Prendo stesso I. Applico trasf. $\perp$ $\hat{\underline{u}} = \underline{Q}\,\underline{u}$



ovvero $\underline{Q}\,\underline{Q}^T = \underline{Q}^T\,\underline{Q} = I$.

Supponiamo x_0 invariato.
Quindi $y = x_0 + \underline{Q}(x - x_0)$

$$(y - x_0) = \underline{Q}(x - x_0)$$

Invece di girare il materiale si è girata la deformazione impressa.
Allora $\hat{\underline{u}}(y) = \underline{Q}\,\underline{u}(x)$

$$\nabla\hat{\underline{u}} = \frac{d\hat{\underline{u}}}{dy} = \frac{d\hat{\underline{u}}}{dx}\,\frac{dx}{dy} = \frac{d(\underline{Q}\,\underline{u})}{dx}\,\frac{dx}{dy}$$

Essendo $\underline{Q}^T(y - x_0) = \underline{Q}^T\,\underline{Q}(x - x_0) \Rightarrow x = x_0 + \underline{Q}^T(y - x_0)$

$$\nabla\hat{\underline{u}} = \underline{Q}\,\nabla\underline{u}\,\underline{Q}^T$$

Allora anche $\boxed{\mathrm{sym}\,\nabla\hat{\underline{u}} = \underline{Q}\,\mathrm{sym}\,\nabla\underline{u}\,\underline{Q}^T} \Rightarrow$

$$\hat{\underline{E}} = \underline{Q}\,\underline{E}\,\underline{Q}^T$$

(34) Nella 2ª deform. si ha.

Poi si associa $\hat{\underline{T}} = \mathbb{C}[\hat{\underline{E}}]$ e $\forall$ direzione si avrà un vettore $\hat{\underline{t}} = \hat{\underline{T}}\,\hat{\underline{n}}$.

Se $\boxed{\hat{\underline{n}} = \underline{Q}\,\underline{n}}$, allora $\boxed{\hat{\underline{t}} = \underline{Q}\,\underline{t}}$. Se $\forall$ punto, $\underline{Q}$ è una trasf. di simmetria.

Comport. del materiale non è distinguibile.

Si ha:

$$\hat{\underline{T}}\,\hat{\underline{n}} = \hat{\underline{t}} = \underline{Q}\,\underline{t} = \underline{Q}\,\underline{T}\,\underline{n} = \boxed{\underline{Q}\,\underline{T}\,\underline{Q}^T\,\hat{\underline{n}} = \hat{\underline{T}}\,\hat{\underline{n}}}$$

ovvero $\underline{T} = \underline{Q}\,\underline{T}\,\underline{Q}^T$. Uguale a $\hat{\underline{E}} = \underline{Q}\,\underline{E}\,\underline{Q}^T$ però solo se ho trasf. di sym (altrimenti $\underline{T}$ è $\neq$).

$\hat{\underline{T}} = \mathbb{C}[\hat{\underline{E}}] = \mathbb{C}[\underline{Q}\,\underline{E}\,\underline{Q}^T]$, $\underline{T} = \mathbb{C}[\underline{E}]$. Invertendo ho $\underline{Q}\,\mathbb{C}[\underline{E}]\,\underline{Q}^T = \mathbb{C}[\underline{Q}\,\underline{E}\,\underline{Q}^T]$. Molt. a sin per $\underline{Q}^T$ e a dx per $\underline{Q}$, ottengo:

$$\boxed{\mathbb{C}[\underline{E}] = \underline{Q}^T\,\mathbb{C}[\underline{Q}\,\underline{E}\,\underline{Q}^T]\,\underline{Q}} \quad (1)$$

L'insieme degli Q è detto

$\mathcal{G}_x$: gruppo delle TRASFORMAZIONI DI SIMMETRIA RISPETTO in x

È un insieme G con definita operaz. "$\circ$" che opera sulle coppie di G con prop. associativa che gode delle seguenti proprietà:

a) i, $i \circ a = a$ $\forall a \in G$ ($\exists$ identità)

b) $\forall a$, $\exists\, a^{-1} \,/\, a^{-1} \circ a = a \circ a^{-1} = i$. ($\exists$ inverso)

IP:

Se $Q \in \mathcal{G}_x \Rightarrow \underline{Q}^T \in \mathcal{G}_x$ (I)

Se $P, Q \in \mathcal{G}_x \Rightarrow (\underline{P}\,\underline{Q}) \in \mathcal{G}_x$ (II)

DIM:

(I). La ① deve valere $\forall\, \underline{E} \in \mathrm{sym}$.

La scriviamo per $\underline{E}' = \underline{Q}^T \underline{E}\, \underline{Q}$

[ricorda: $(\underline{A}\,\underline{B})^T = \underline{B}^T \underline{A}^T$]

$$(\underline{Q}^T \underline{E}\, \underline{Q})^T = \underline{Q}^T \underline{E}^T (\underline{Q}^T)^T = \underline{Q}^T \underline{E}\, \underline{Q} \quad \checkmark \quad \underline{E} \text{ è sym}$$

$$\mathbb{C}[\underline{Q}^T \underline{E}\, \underline{Q}] = \underline{Q}^T \mathbb{C}[\underline{Q}\,\underline{Q}^T \underline{E}\, \underline{Q}\,\underline{Q}^T]\, \underline{Q}$$

$$\mathbb{C}[\underline{Q}^T \underline{E}\, \underline{Q}] = \underline{Q}^T \mathbb{C}[\underline{E}]\, \underline{Q} \quad \text{premolt } \underline{Q} \text{ e molt } \underline{Q}^T.$$

$$\underline{Q}\, \mathbb{C}[\underline{Q}^T \underline{E}\, \underline{Q}]\, \underline{Q}^T = \mathbb{C}[\underline{E}] \quad \text{C.V.D.}$$

~~(II). Ora prendiamo $\underline{E}' = \underline{Q}\,\underline{E}\,\underline{Q}^T$ e risolvo ①:~~

~~$\mathbb{C}[\underline{Q}\,\underline{E}\,\underline{Q}^T] = \underline{Q}^T$~~

Scriviamo ① per $\underline{P}$:

$$\mathbb{C}[\underline{E}] = \underline{P}^T \mathbb{C}[\underline{P}\,\underline{E}\,\underline{P}^T]\, \underline{P} \quad \text{Prendiamo } \underline{Q}\,\underline{E}\,\underline{Q}^T.$$

$$\mathbb{C}[\underline{Q}\,\underline{E}\,\underline{Q}^T] = \underline{P}^T \mathbb{C}[\underline{P}\,\underline{Q}\,\underline{E}\,\underline{Q}^T \underline{P}^T]\, \underline{P}$$

Da ①, premolt per $\underline{Q}$ e molt. per $\underline{Q}^T$, ho:

$$\mathbb{C}[\underline{Q}\,\underline{E}\,\underline{Q}^T] = \underline{Q}\, \mathbb{C}[\underline{E}]\, \underline{Q}^T = \underline{P}^T \mathbb{C}[\underline{P}\,\underline{Q}\,\underline{E}\,\underline{Q}^T \underline{P}^T]\, \underline{P}.$$

Premolt. per $\underline{Q}^T$ e molt per $\underline{Q}$

$$\underline{Q}^T \underline{P}^T \mathbb{C}[\underline{P}\,\underline{Q}\,\underline{E}\,\underline{Q}^T \underline{P}^T]\, \underline{P}\,\underline{Q} = \mathbb{C}[\underline{E}] \quad \text{C.V.D.}$$

$$\Downarrow$$

Le simmetrie sono un gruppo di trasf. ortogon.

Le trasf. ortog. sono di 2 tipi.

(36) $\underline{Q}^T \underline{Q} = \underline{I}$

$(\det \underline{Q})^2 = 1$ Quindi $\det \underline{Q} = \pm 1$.

- Se $\det \underline{Q} = 1$ la trasf. sitop. è una ROTAZIONE

$\mathcal{G}_x$ contiene sempre $\underline{I}$ e $-\underline{I}$.

L'insieme di tutti i tensori $\perp$ si chiama Orth.

" " " " " " $\perp$ di rot. si " $\mathrm{Orth}^+ = \mathrm{Rot}$.

- Ogni $\underline{Q}$ con $\det \underline{Q} = -1$ si può sempre scrivere come $-\underline{I}\,\underline{R}$.

Quindi $\mathcal{G}_x = \left\{ -\underline{I}, \underline{I} \right\} \times \mathcal{G}_x^+$

Le 21 comp. di $\mathbb{C}$ si tabellano: (caso + generale)

$$\begin{bmatrix}
\mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \mathbb{C}_{1112} & \mathbb{C}_{1123} & \mathbb{C}_{1131} \\
 & \mathbb{C}_{2222} & \mathbb{C}_{2233} & \mathbb{C}_{2212} & \mathbb{C}_{2223} & \mathbb{C}_{2231} \\
 & & \mathbb{C}_{3333} & \mathbb{C}_{3312} & \mathbb{C}_{3323} & \mathbb{C}_{3331} \\
 & & & \mathbb{C}_{1212} & \mathbb{C}_{1223} & \mathbb{C}_{1231} \\
 & & & & \mathbb{C}_{2323} & \mathbb{C}_{2331} \\
 & & & & & \mathbb{C}_{3131}
\end{bmatrix}$$

$$T_{11} = \mathbb{C}_{1111} E_{11} + \mathbb{C}_{1122} E_{22} + \mathbb{C}_{1133} E_{33} + \mathbb{C}_{1112} \overset{(\text{sono } =)}{(E_{12} + E_{21})} +$$

$$+ 2\,\mathbb{C}_{1123} E_{23} + 2\,\mathbb{C}_{1131} E_{31}$$

Si ha:

$$\mathbb{C}[\underline{A}] \cdot \underline{B} = \mathbb{C}[\mathrm{sym}\,\underline{A}] \cdot \mathrm{sym}\,\underline{B}$$

$$\mathbb{C}[\underline{A}] = \mathbb{C}[\mathrm{sym}\,\underline{A} + \mathrm{skw}\,\underline{A}] = \mathbb{C}[\mathrm{sym}\,\underline{A}] + \cancel{\mathbb{C}[\mathrm{skw}\,\underline{A}]}$$

$$\mathbb{C}[\mathrm{sym}\,\underline{A}] \cdot \underline{B} = \mathbb{C}[\mathrm{sym}\,\underline{A}] \cdot (\mathrm{sym}\,\underline{B} + \mathrm{skw}\,\underline{B})$$

Il p.s. tra sym e skw è $= 0$

Allora:

$$\mathbb{C}[\underline{m} \otimes \underline{n}] \cdot (\underline{a} \otimes \underline{b}) = \mathbb{C}[\mathrm{sym}(\underline{m} \otimes \underline{n})] \cdot \mathrm{sym}(\underline{a} \otimes \underline{b})$$

Sappiamo che $\mathrm{sym}(\underline{u} \otimes \underline{v}) = \frac{1}{2}(\underline{u} \otimes \underline{v} + \underline{v} \otimes \underline{u})$

Se abbiamo $\underline{Q}(\underline{m} \otimes \underline{n})\underline{Q}^T$ allora $= \underline{Q}\,\underline{m} \otimes \underline{Q}\,\underline{n}$. Dim:

$$(\underline{Q}(\underline{m} \otimes \underline{n})\underline{Q}^T)\,\underline{a} = \underline{Q}(\underline{m} \otimes \underline{n})(\underline{Q}^T \underline{a}) \text{ [componiamo tensori]} =$$

$$= \underline{Q}\,\underline{m}(\underline{n} \cdot \underline{Q}^T \underline{a}) = \underline{Q}\,\underline{m}(\underline{Q}\,\underline{n} \cdot \underline{a}) = (\underline{Q}\,\underline{m} \otimes \underline{Q}\,\underline{n})\,\underline{a}$$

CVD

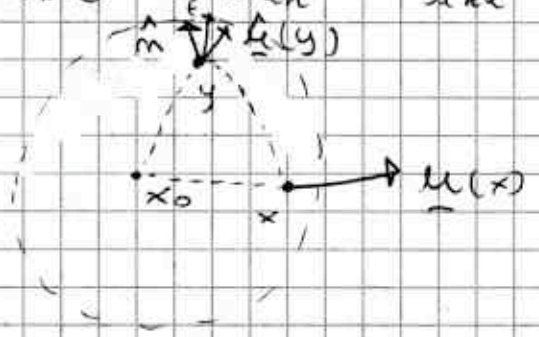
Allora:

$$\underline{Q}\,\mathrm{sym}(\underline{m} \otimes \underline{n})\,\underline{Q}^T = \mathrm{sym}(\underline{Q}\,\underline{m} \otimes \underline{Q}\,\underline{n}).$$

17/3/09

Eq. costit. $\underline{T} = \mathbb{C}[\underline{E}]$, $\mathbb{C}$ tens. di elasticità

$\mathbb{C}_{ijkl} = \mathbb{C}_{ijlk} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}$, elasticità



$$y = x_0 + \underline{Q}(x - x_0)$$

$$\hat{\underline{u}}(y) = \underline{Q}\,\underline{u}(x)$$

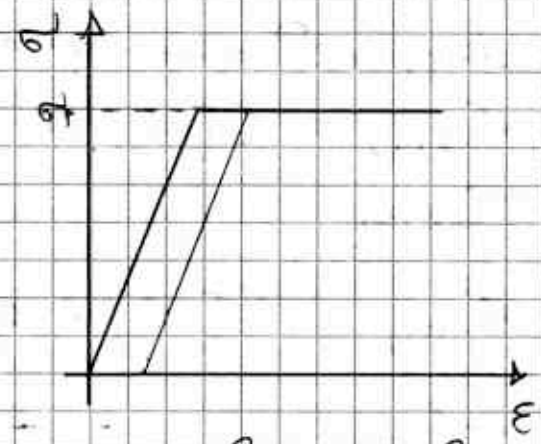
$$\hat{\underline{E}} = \underline{Q}\,\underline{E}\,\underline{Q}^T, \text{ se } \forall x \text{ e } \underline{m} \quad \text{IP:}$$

$$\hat{\underline{t}}(y, \hat{\underline{m}}) = \underline{Q}\,\underline{t}(x, \underline{m}), \ \forall \underline{E} \in \mathrm{Sym}$$

$\underline{Q}$ è una trasf. di simmetria per il materiale

Materiale è elastico quando sforzo è solo dal valore istante della deformazione e non dalla

una storia.



Corr. Rotazione intorno a x_0.

Il campo di spostam $\hat{\underline{u}}(y)$ corrisp. $\hat{\underline{E}}$.

L'ip. fatta conduce a $\hat{\underline{T}} = \underline{Q}\,\underline{T}\,\underline{Q}^T$

e con rel. costitutiva si ha

$\mathbb{C}[\underline{E}] = \underline{Q}^T\,\mathbb{C}[\underline{Q}\,\underline{E}\,\underline{Q}^T]\,\underline{Q}$, $\forall\, \underline{E} \in \mathrm{Sym}$ Gruppo $\mathcal{G}$.

$\mathcal{G} = \{-\underline{I}, \underline{I}\} \times \mathcal{G}^+$.

$* \; \mathbb{C}[\underline{E}] = \underline{Q}^T\,\mathbb{C}[\underline{Q}\,\underline{E}\,\underline{Q}^T]\,\underline{Q}$

$* \; \mathbb{C}[\underline{A}] \cdot \underline{B} = \mathbb{C}[\mathrm{sym}\,\underline{A}] \cdot \mathrm{sym}\,\underline{B}$

$* \; \underline{A}(\underline{m} \otimes \underline{n})\,\underline{B} = \underline{A}\,\underline{m} \otimes \underline{B}^T\,\underline{n}$

$\mathbb{C}[\underline{a} \otimes \underline{b}] \cdot (\underline{m} \otimes \underline{n}) = \mathbb{C}[\mathrm{sym}(\underline{a} \otimes \underline{b})] \cdot \mathrm{sym}(\underline{m} \otimes \underline{n}) =$

$= \underline{Q}^T\,\mathbb{C}[\underline{Q}\,\mathrm{sym}(\underline{a} \otimes \underline{b})\,\underline{Q}^T]\,\underline{Q} \cdot \mathrm{sym}(\underline{m} \otimes \underline{n}) =$

$= \quad \mathbb{C}[\quad '' \quad] \;''\; \cdot \;''\; \underline{Q}^T =$

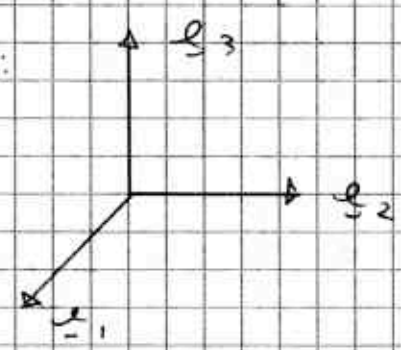
$= \mathbb{C}[\mathrm{sym}(\underline{Q}\underline{a} \otimes \underline{Q}\underline{b})] \cdot \mathrm{sym}(\underline{Q}\,\underline{m} \otimes \underline{Q}\,\underline{n})$

$\Downarrow$

$* \; \mathbb{C}[\underline{a} \otimes \underline{b}] \cdot (\underline{m} \otimes \underline{n}) = \mathbb{C}[\underline{Q}\underline{a} \otimes \underline{Q}\underline{b}] \cdot (\underline{Q}\,\underline{m} \otimes \underline{Q}\,\underline{n})$

$\underline{R}^{\varphi}_{\underline{e}}$: rotaz. di ampiezza φ intorno ad asse $\underline{e}$

Supponiamo $\mathcal{G}^+ \ni \underline{R}^{\pi/2}_{\underline{e}_3}$ allora:



$\underline{R}\,\underline{e}_1 = \underline{e}_2$; $\underline{R}\,\underline{e}_2 = -\underline{e}_1$

$\underline{R}\,\underline{e}_3 = \underline{e}_3$

Applichiamo quest' ultima propr.

$$\mathbb{C}_{1111} = \mathbb{C}[\underline{e}_1 \otimes \underline{e}_1]\cdot(\underline{e}_1 \otimes \underline{e}_1) = \mathbb{C}[\underline{e}_2 \otimes \underline{e}_2]\cdot(\underline{e}_2 \otimes \underline{e}_2) = \mathbb{C}_{2222}$$

$$\mathbb{C}_{1112} = \mathbb{C}[\underline{e}_1 \otimes \underline{e}_2]\cdot(\underline{e}_1 \otimes \underline{e}_1) = \mathbb{C}[\underline{e}_2 \otimes (-\underline{e}_1)]\cdot(\underline{e}_2 \otimes \underline{e}_2) = -\mathbb{C}_{2221}$$

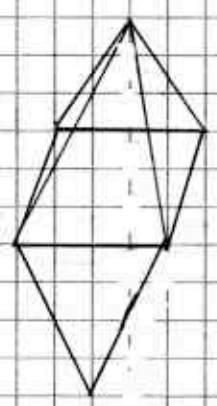
$\mathbb{C}_{3312} = -\mathbb{C}_{3321} = 0$ per simm. generale sui 2 indici

$\Downarrow$

gruppo delle sym comprende le $R_{\underline{e}_3}^{\frac{\pi}{2}}$

$\exists$ materiali ANISOTROPI

In particolare i materiali CRISTALLINI.

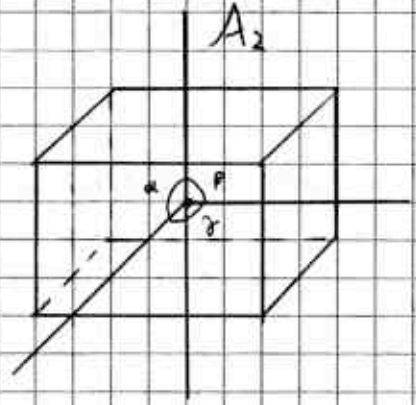


Cristallo elementare. Se $\exists$ retta che fa ruotare di $\varphi = \frac{2\pi}{n}$ il cristallo appare identico e la retta è detta <u>asse di simmetria</u>.

Un materiale del sistema <u>TRICLINO</u> non ha assi di sym.

Ha tutte e 21 le elasticità distinte

Un mat. del sist. <u>MONOCLINO</u> ha asse di sym binario (A_2)



$\alpha = \beta = \frac{\pi}{2}$

$\gamma \neq \frac{\pi}{2}$

Se ruota di π intorno ad A_2 rimane identico.

Ha il gruppo $\mathcal{G}^+$ generato da $\underline{R}_{\underline{e}_3}^{\pi}$.

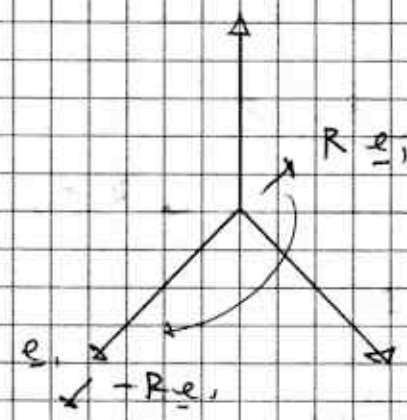
$$\mathcal{G} = \{-\underline{I}, \underline{I}\} \times \{\underline{R}_{\underline{e}_3}^{\pi}\}$$

Conv. $\underline{R} = -\underline{I}\ \underline{R}^{\pi}_{\underline{e}_3}$

$\underline{Q}\,\underline{e}_1 = \underline{e}_1$

$\underline{Q}\,\underline{e}_2 = \underline{e}_2$

$\underline{Q}\,\underline{e}_3 = -\underline{e}_3$



$\Downarrow$

$$\mathbb{C}_{ijkl} = \mathbb{C}\left[\underline{e}_k \otimes \underline{e}_l\right]\cdot\left(\underline{e}_i \otimes \underline{e}_j\right) = \mathbb{C}\left[\underline{Q}\,\underline{e}_k \otimes \underline{Q}\,\underline{e}_l\right]\cdot\left(\underline{Q}\,\underline{e}_i \otimes \underline{Q}\,\underline{e}_j\right)$$

Allora $\mathbb{C}_{1123} = -\mathbb{C}_{1123} = 0$. Succede ogni volta che c'è un solo 3 negli indici o 3 volte (- - - -).

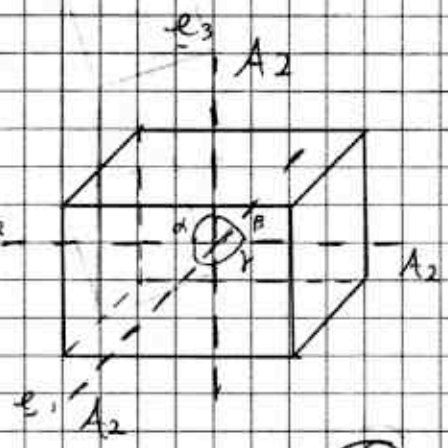
Mat. monoclino ha 13 componenti

$$\begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \mathbb{C}_{1112} & 0 & 0 \\ & \mathbb{C}_{2222} & \mathbb{C}_{2233} & \mathbb{C}_{2212} & 0 & 0 \\ & & \mathbb{C}_{3333} & \mathbb{C}_{3312} & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \\ & & & & \mathbb{C}_{2323} & \mathbb{C}_{2331} \\ & & & & & \mathbb{C}_{3131} \end{bmatrix}$$

Materiali <u>ORTOTROPI</u>:

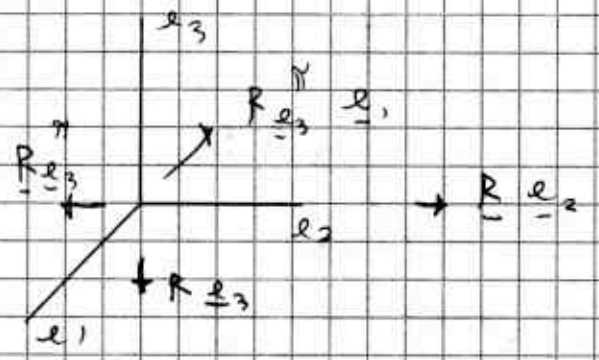
- Sistema <u>ROMBICO</u>: presenta 3 A_2

Parallelepipedo rettangolo, $\alpha = \beta = \gamma = \frac{\pi}{2}$



$$\mathcal{G} = \left\{-\underline{I}, \underline{I}\right\} \times \left\{\underline{R}^{\pi}_{\underline{e}_3}, \underline{R}^{\pi}_{\underline{e}_1}\right\}$$

Supercháma $\underline{R}^{\pi}_{\underline{e}_3}$



Per $R_{e_1}^{\pi}$

$$R = R_{e_1}^{\pi} R_{e_2}^{\pi} = R_{e_2}^{\pi}$$

Comp. $Q = -I \, R_{e_1}^{\pi}$

$\hookrightarrow e_1 \to -e_1$; $e_2 = e_2$, $e_3 = e_3$

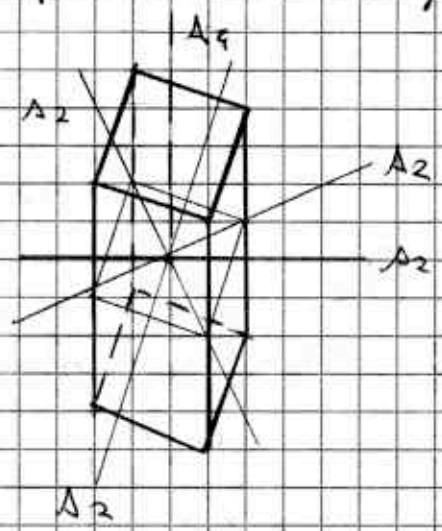
$C_{1112} = -C_{1112} = 0$

Nulle se indice hanno o 3 o 1 "uno". 9 comp.

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & & & & C_{2323} & 0 \\ & & & & & C_{3131} \end{bmatrix}$$

- Sistema TETRAGONALE: A_4, $4A_2$

Ex: prisma rettang. a sezione quadrata.



$$g = \left\{ -I, I \right\} \times \left\{ R_{e_3}^{\frac{\pi}{2}}, R_{e_1}^{\pi} \right\}$$

$C_{1111} = C_{2222}$

Infatti $R_{e_3}^{\frac{\pi}{2}} e_1 = e_2$,

$R_{e_3}^{\frac{\pi}{2}} e_2 = -e_1$

$C_{1133} = C_{2233}$; $C_{1313} = C_{2323}$

Ora 6 elast. indipendenti

$$\begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & 0 & 0 & 0 \\ & \mathbb{C}_{1111} & \mathbb{C}_{1133} & 0 & 0 & 0 \\ & & \mathbb{C}_{3333} & 0 & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \\ & & & & \mathbb{C}_{2323} & 0 \\ & & & & & \mathbb{C}_{2323} \end{bmatrix}$$

- materiale TRASVERSALMENTE ISOTROPO

sostituisce

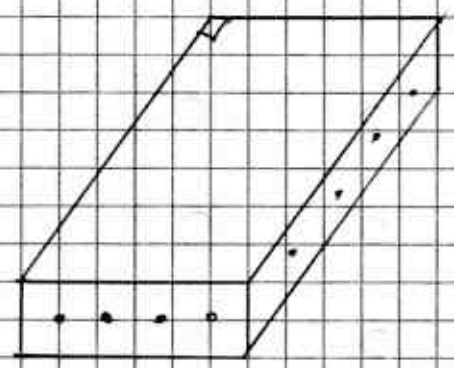
$\underset{\underline{e}_3}{\overset{\varphi}{R}}$, $\forall \varphi$; $\mathbb{C}_{1212} = \dfrac{\mathbb{C}_{1111} - \mathbb{C}_{1122}}{2}$

- un materiale ISOTROPO ha $\mathcal{G} = \{-\underline{\underline{I}}, \underline{\underline{I}}\} \times Rot$

$$\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ & & 2\mu+\lambda & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}$$

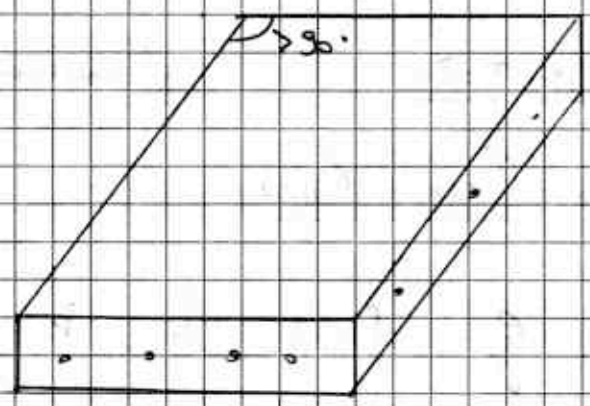
$\mathbb{H}$

Cons. piastra rettangolare in c.a. con armatura longit. e trasversale



[ricorda: cristalli ripetuti ⇒ stessa forma e regole nel grande]

Scriveremo eq del mat. ort. romb. (anche se acciaio distribuito in modo diverso). Se stessa armatura è tetr.



Piastra in c.a. mat. monoclino

Tipico dei compositi moderni

H

FORMA CLASSICA PROBLEMA D'EQUILIBRIO

$$\begin{cases} \operatorname{div} \underline{T} + \underline{b} = \underline{0} & \text{in } C \\ \underline{u} = \hat{\underline{u}} & \text{su } \partial C_1 \\ \underline{T}\,\underline{n} = \hat{\underline{t}} & \text{su } \partial C_2 \end{cases}$$

Rel costitutiva: $\underline{T} = \mathbb{C}[\nabla \underline{u}]$

$$\begin{cases} \operatorname{div} \mathbb{C}[\nabla \underline{u}] + \underline{b} = \underline{0} & \text{in } C \\ u = \hat{\underline{u}} & \text{su } \partial C_1 \\ \mathbb{C}[\nabla \underline{u}]\,\underline{n} = \hat{\underline{t}} & \text{su } \partial C_2 \end{cases}$$

→ equil. in termini di spost.

Vale la sovrapp. effetti:

$\underline{u}_1$ soluzione da $\underline{b}_1, \hat{\underline{u}}_1, \hat{\underline{t}}_1$ (stati)

$\underline{u}_2$ " " $\underline{b}_2, \hat{\underline{u}}_2, \hat{\underline{t}}_2$ "

ovvero $\lambda_1 \underline{u}_1 + \lambda_2 \underline{u}_2$ è soluz.

$\lambda_1 \underline{b}_1 + \lambda_2 \underline{b}_2$, $\lambda \hat{\underline{u}}_1 + \lambda \hat{\underline{u}}_2$, $\lambda_1 \hat{\underline{t}}_1 + \lambda_2 \hat{\underline{t}}_2$

(valgono le comb. lineari)

IP:

$$\operatorname{div} \mathbb{C}\left[\nabla(\lambda_1 \underline{u}_1 + \lambda_2 \underline{u}_2)\right] + \underline{b}_1 + \underline{b}_2 = \underline{0}$$

DIM: div e ∇ sono operat. lineari. Ho:

$$\lambda_1 \operatorname{div} \cancel{\mathbb{C}[\nabla \underline{u}]} + \lambda_2 \operatorname{div} \cancel{\mathbb{C}[\nabla \underline{u}_2]} + \cancel{\underline{b}_1} + \cancel{\underline{b}_2} = \underline{0} \quad \text{CVD}$$

Su $\partial\Omega_2$: $\mathbb{C}\left[\nabla(\lambda_1 \underline{u}_1 + \lambda_2 \underline{u}_2)\right] \underline{n} = \left(\lambda_1 \mathbb{C}[\nabla \underline{u}_1] + \lambda_2 \mathbb{C}[\nabla \underline{u}_2]\right) \underline{n} = \lambda_1 \hat{\underline{t}}_1 + \lambda_2 \hat{\underline{t}}_2$. CVD.

Si fa anche l'ipotesi $\mathbb{C}[\underline{E}] \cdot \underline{E} > 0$, $\forall \underline{E} \in \mathrm{Sym}$, $\underline{E} \neq \underline{0}$.
(è sicur. positivo $\Rightarrow$ unicità soluz. prob. equilibrio).

$$\begin{cases} \operatorname{div} \mathbb{C}[\nabla \underline{u}] + \underline{b} = \underline{0} & \text{in } \Omega \\ \underline{u} = \hat{\underline{u}} & \text{su } \partial\Omega_1 \\ \mathbb{C}[\nabla \underline{u}]\, \underline{n} = \hat{\underline{t}} & \text{su } \partial\Omega_2 \end{cases}$$

IP:
Sol. è unica a meno di spost. rigido.

DIM:
Supponiamo $\exists$ per assurdo 2 $\underline{u}_1, \underline{u}_2$.
Prendiamo $\underline{u} = \underline{u}_1 - \underline{u}_2$.
Per soluz. effetti $\underline{u}$ è sol per gli stessi

dati sia su $\underline{u}_1$ che su $\underline{u}_2$.

Quindi $\underline{u}$ è sol. di $\underline{b} = 0$, $\hat{\underline{u}} = 0$ su ∂C_1, $\hat{\underline{t}} = 0$ su ∂C_2

Allora div $\underline{T} = \underline{0}$ in C.

Moltiplico scalarm. per $\underline{u}$: $0 = (\operatorname{div} \underline{T}) \cdot \underline{u}$.

Identità differenziale:

$$\underline{\operatorname{div} \underline{T} \cdot \underline{u} = \operatorname{div}(\underline{T}^T \underline{u}) - \underline{T} \cdot \underline{\nabla u}}$$

$$\underline{T}_{,i} \underline{e}_i \cdot \underline{u} = (\underline{T} \underline{e}_i \cdot \underline{u})_{,i} - \underline{T} \underline{e}_i \cdot \underline{u}_{,i}$$

$$\operatorname{div} \underline{T} \cdot \underline{u} = \operatorname{div} \underline{T}^T \underline{u} - \underline{T} \cdot \nabla \underline{u}$$

$$\left[\underline{T} \underline{e}_i \cdot \underline{u} = \underline{e}_i \cdot \underline{T}^T \underline{u} \; ; \; (\underline{T} \underline{e}_i \cdot \underline{u})_{,i} = (\underline{e}_i \cdot \underline{T}^T \underline{u})_{,i} = \right.$$
$$\left. = \underline{e}_i \cdot (\underline{T}^T \underline{u})_{,i} \right]$$

Quindi $0 = (\operatorname{div} \underline{T}) \cdot \underline{u} = \operatorname{div} \underline{T}^T \underline{u} - \underline{T} \nabla \underline{u}$. Integro:

$$0 = \int_C (\operatorname{div} \underline{T}) \cdot \underline{u} = \int_C \operatorname{div} \underline{T}^T \underline{u} - \int_C \underline{T} \cdot \nabla \underline{u}$$

Applichiamo th. divergenza:

$$= \int_{\partial C} \cancel{\underline{T}^T \underline{u} \cdot \underline{n}} \; \underline{T} \underline{n} \cdot \underline{u} - \int_{\partial C} \underline{T} \cdot \nabla \underline{u} =$$

Su ∂C_1 è nullo $\underline{u}$

" ∂C_2 è " $\underline{T} \underline{n}$

$$= -\int_C \mathbb{C}[\nabla \underline{u}] \cdot \nabla \underline{u} = -\int_C \mathbb{C}[\underline{\varepsilon}] \cdot \underline{\varepsilon}$$

. Deve essere $\underline{\varepsilon} = \underline{0}$, ma $\Rightarrow$ se $\operatorname{sym} \nabla (\underline{u}_1 - \underline{u}_2) = \underline{0}$ ho solo sol!

$\Downarrow$

2 sol possono differire solo per spostamento rigido.

CVD

Lavoro:

$$L = \int_C \underline{b} \cdot \underline{u} + \int_{\partial C} \underline{t} \cdot \underline{u}$$

compiuto da $\underline{b}$ e $\underline{t}$ per effetto di $\underline{u}$ non è il lav. di deform.

TEOREMA DI BETTI

Campo spost. $\underline{u}_1$ in equil. con $\underline{b}_1$:

$$\operatorname{div} \mathbb{C}[\nabla \underline{u}_1] + \underline{b}_1 = \underline{0} \quad \text{in } C$$

$$\mathbb{C}[\nabla \underline{u}_1] = \underline{t}_1 \quad \text{su } \partial C$$

Campo spost. $\underline{u}_2$ in equil. con $\underline{b}_2$:

$$\operatorname{div} \mathbb{C}[\nabla \underline{u}_2] + \underline{b}_2 = 0 \quad \text{in } C$$

$$\mathbb{C}[\nabla \underline{u}_2] = \underline{t}_2 \quad \text{in } \partial C$$

Teo:

$$\boxed{L_{12} = \int_C \underline{b}_1 \cdot \underline{u}_2 + \int_{\partial C} \underline{t}_1 \cdot \underline{u}_2 = \int_C \underline{b}_2 \cdot \underbrace{\underline{u}_1}_{L_{21}} + \int_{\partial C} \underline{t}_2 \cdot \underline{u}_1 \quad \sigma = \mathbb{C}^T}$$

Dim:

$$L_{12} = \int_C \underline{b}_1 \cdot \underline{u}_2 + \int_{\partial C} \underline{T}_1 \underline{n} \cdot \underline{u}_2 = \int_C \underline{b}_1 \cdot \underline{u}_2 + \int_{\partial C} \underline{T}_1^T \underline{u}_2 \cdot \underline{n} =$$

$$= \int_C \left(\underline{b}_1 \cdot \underline{u}_2 + \operatorname{div}\left(\underline{T}_1^T \underline{u}_2 \right) \right) = \int_C \left(\underline{b}_1 \cdot \underline{u}_2 + \operatorname{div} \underline{T}_1 \cdot \underline{u}_2 + \right.$$

$$\left. T_1 \cdot \nabla u_2 \right) = \int_C \mathbb{C}[\nabla \underline{u}_1] \cdot \nabla \underline{u}_2 = \int_C \nabla \underline{u}_1 \cdot \mathbb{C}[\nabla \underline{u}_2] =$$

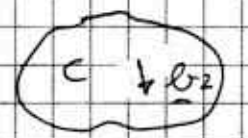
$$= \int_C \nabla \underline{u}_1 \cdot \underline{T}_2 = \int_C \left(\operatorname{div}\left(\underline{T}_2^T \underline{u}_1 \right) - \operatorname{div} \underline{T}_2 \cdot \underline{u}_1 \right) =$$

$$= \int_{\partial C} \underline{T}_2 \, \underline{n} \cdot \underline{u}_1 \; (+) \int_C \underline{b}_2 \cdot \underline{u}_1 = L_{21}$$

$[\operatorname{div} \underline{T}_2 = -\underline{b}_2]$

23/03/09

$$\operatorname{div} \mathbb{C}[\nabla \underline{u}_1] + \underline{b}_1 = \underline{0}, \quad \text{in } C$$
$$\mathbb{C}[\nabla \underline{u}_1]\,\underline{n} = \underline{t}_1, \quad \text{su } \partial C \qquad \text{IP: } \mathbb{C} = \mathbb{C}^T$$



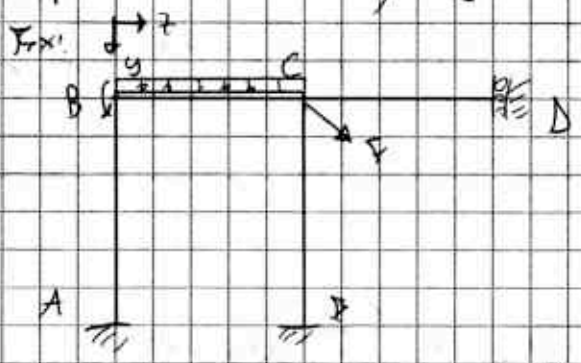
$$\operatorname{div} \mathbb{C}[\nabla \underline{u}_2] + \underline{b}_2 = \underline{0}, \quad \text{in } C$$
$$\mathbb{C}[\nabla \underline{u}_2]\,\underline{n} = \underline{t}_2, \quad \text{su } \partial C$$

$$\int_C \underline{b}_1 \cdot \underline{u}_2 + \int_{\partial C} \underline{t}_1 \cdot \underline{u}_2 = \int_C \underline{b}_2 \cdot \underline{u}_1 + \int_{\partial C} \underline{t}_2 \cdot \underline{u}_1$$

IP: Da questa discende la sym della matrice di rigid.

Dim: Nel metodo degli s. ho spost. nodali incogniti ed esprimo $\boxed{\underline{K}\,\underline{s} = \underline{f}}$ con $\underline{K}$: matr. di rig., $\underline{s}$: vett. spost. nodali, $\underline{f}$: vett. termine noto, d'carichi

Es:



$$\underline{s} = \begin{bmatrix} v_B \\ w_B \\ \varphi_D \\ v_C \\ w_C \\ \varphi_C \\ v_D \end{bmatrix}, \quad \underline{K} : 7 \times 7, \quad \underline{f} \text{ d'carichi su aste e nodi}$$

$\underline{K}$ in componenti: $\sum_{j=1}^{n} K_{ij} s_j = f_i, \quad i = 1 \dots n$

Supponiamo solo carichi nodali.

Supponiamo inoltre $s_k = 1$, $s_j = 0$, $j \neq k$.
$K_{ik} = f_i$, $i = 1 \dots n$ } sist. ①

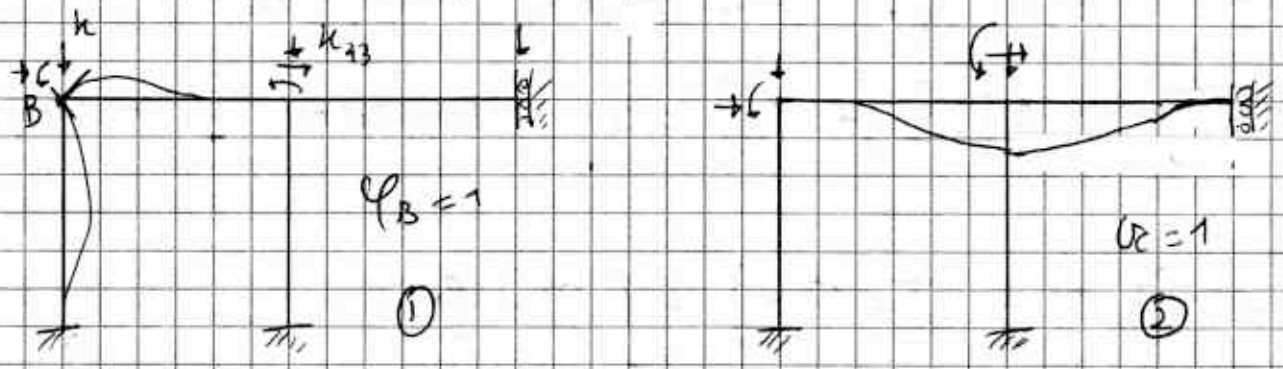
Scegliamo ora $s_h = 1$, $s_j = 0$, $j \neq h$
$K_{ih} = f_i$, $i = 1 \dots n$ } sist. ②

Dopo teorema:

$$L_{12} = f_i^{(1)} \cdot \delta_{ik} = L_{21} = f_i^{(2)} \delta_{ik} \quad \text{ovvero} \quad f_k^{(1)} = f_k^{(2)}$$

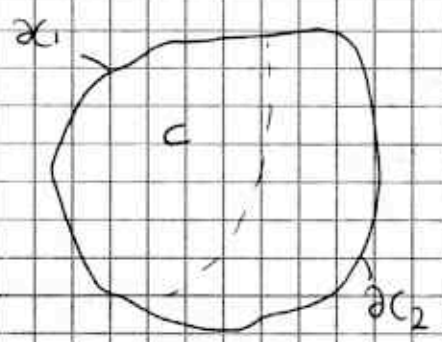
quindi $\boxed{k_{hk} = k_{kh}}$ ⇒ matrice k è simm. CVD

Ex:



$$\begin{bmatrix} v_B \\ u_B \\ \varphi_B \\ v_C \\ u_C \\ \varphi_C \\ v_D \end{bmatrix}$$

$$L_{12} = k_{43} \cdot 1$$
$$L_{21} = k_{34} \cdot 1$$



$$\operatorname{div} \underline{T} + \underline{b} = \underline{0}, \quad \text{in } C$$
$$\underline{u} = \hat{\underline{u}} \quad \text{su } \partial C_1$$
$$\underline{T}\,\underline{n} = \hat{\underline{t}}, \quad \text{su } \partial C_2$$
$$\underline{T} = \mathbb{C}[\nabla \underline{u}]$$

Ma non è adatta se usata con metodi approx o metodi FEM a voglio formulaz diverse.

FORMA VARIAZIONALE

$$\boxed{\int_C \underline{T} \cdot \nabla \underline{v} \, dV - \int_C \underline{b} \cdot \underline{v} \, dV - \int_{\partial C_2} \hat{\underline{t}} \cdot \underline{v} \, da = 0} \rightarrow \text{FO VARIAZIONALE}$$

$\underline{T} = \mathbb{C}[\nabla \underline{u}]$. Incognita è sempre $\underline{u}$.

Si deve trovare $\underline{u}$ / $\underline{u} = \hat{\underline{u}}$ su ∂C_1

$\forall\ \underline{v} \ / \ \underline{v} = \underline{0}$ su ∂C_1 verifica l'eq. variazion.

IP:
Se $\underline{u}$ è sol del prob. in forma classica lo è anche x variaz.
$\operatorname{div} \sigma[\nabla \underline{u}] + \underline{b} = 0$. Molt. scalarmente per $\underline{v} = \underline{0}$ su ∂C_1, poi sotto traccia. Int. su C:

$$\int_C \underline{v} \cdot \left(\operatorname{div} \sigma[\overset{T}{\nabla \underline{u}}] + \underline{b} \right) = 0 \qquad \text{Applico th. diverg.} =$$

$$= \int_C \left(\operatorname{div}(\underline{T}^T \underline{v}) - \underline{T} \cdot \nabla \underline{v} \right) + \int_C \underline{v} \cdot \underline{b} = 0 \quad ; \quad // \ //$$

$$\left[\operatorname{div}(\underline{T}^T \underline{v}) = \operatorname{div} \underline{T} \cdot \underline{v} + \underline{T} \cdot \nabla \underline{v} \right]$$

$$= \int_{\partial C} \underline{T}^T \underline{v} \cdot \underline{n} \, da - \int_C \left(\underline{T} \cdot \nabla \underline{v} - \underline{b} \cdot \underline{v} \right) = 0$$

pari a $\underline{T}\,\underline{n} \cdot \underline{v}$
(per sfg. di trasposto)

Su ∂C_1 $\underline{v} = 0$, rimane int. su ∂C_2.
Su ∂C_2, $\underline{T}\,\underline{n} = \hat{\underline{t}}$, quindi:

$$\int_{\partial C_2} \hat{\underline{t}} \cdot \underline{v} \, da - \int_C \left(\underline{T} \cdot \nabla \underline{v} - \underline{b} \cdot \underline{v} \right) = 0 \quad = \text{eq. variaz. a - del segno}$$

$$\Downarrow$$

Ogni sol del prob. classico è sol del problema variazionale. CVS

Sappiamo che $\underline{T} = \sigma[\nabla \underline{u}]$ nel prob. classico.
In componenti: $T_{ij} = \sigma_{ijhk} \, u_{h,k}$

$$(\operatorname{div} \underline{T})_i = T_{ij,j} = \sigma_{ijhk} \, u_{h,kj}$$

(Bo) $\underline{u}$ deve essere di classe C^2 in C $[\underline{u} \ C^2(C)]$

$\underline{u}$ deve essere di classe C^1 su ∂C $\hookrightarrow$ $\underline{u} \begin{cases} C^2(C) \\ C^1(\partial C) \end{cases}$

PROPRIETÀ DI REGOLARITÀ

Per il prob. variazionale si ha $\underline{u} \begin{cases} C^1(C) \\ C^0(\partial C) \end{cases}$
$\underline{v}$
[non si sono derivate 2]

In variaz. si possono risolvere prob. + generali
Cons. sol. valide anche per prob. classici.

Verifica:

Int. per part.:

$$\underline{T} \cdot \nabla \underline{v} = \operatorname{div}(\underline{T}^T \underline{v}) - \operatorname{div} \underline{T} \cdot \underline{v}$$

$$\int_C \operatorname{div}(\underline{T}^T \underline{v}) - \int_C (\operatorname{div} \underline{T} + \underline{b}) \cdot \underline{v} - \int_{\partial C_2} \hat{\underline{t}} \cdot \underline{v} = 0$$

tel div.

$$\int_{\partial C_2} \underline{T}^T \underline{v} \cdot \underline{n} = \underline{T}\underline{n} \cdot \underline{v}$$

Dall'eq. variaz. si ha quindi:

$$-\int_C (\operatorname{div} \underline{T} + \underline{b}) \cdot \underline{v} \, dv + \int_{\partial C_2} (\underline{T}\underline{n} - \hat{\underline{t}}) \cdot \underline{v} \, da = 0$$

Essendo $\underline{v}$ arbitraria, con $\underline{v} = \underline{0}$ su ∂C_2

$$\int_C (\operatorname{div} \underline{T} + \underline{b}) \cdot \underline{v} = \underline{0}$$

, supponiamo $(\operatorname{div} \underline{T} + \underline{b})_i(x) > 0$
[comp. i-esima non nulla nel punto x]

Ip. è $\underline{u}$ $C^2(C)$

Per norman. segno f continua,
$\exists$ I in cui f è sempre >0

Essendo $\underline{v}$ arb., prendiamo
$v_i > 0$ in I e nulla nel resto del corpo.

Si ha quindi:

$$\int_I (\operatorname{div} \underline{T} + \underline{b})_i \, v_i = 0 \Rightarrow \underline{\operatorname{div} \underline{T} + \underline{b} = 0 \text{ in } C}$$

Idem su ∂C_2: $\underline{T}\underline{n} = \hat{\underline{t}}$ su ∂C_2 scelta arbitrariamente di $\underline{v}$.

$\Downarrow$

Sol. prob. var. è sol. prob. forma classica CVS
↳ con soluz. regol.

- C. al bordo si dicono ESSENZIALI se cons. nell'assegnazione della f. incognita, nella forma in cui la variazione appare nell'int. di bordo dell'eq. variaz.
  Qui sono l'assegnaz. di $\underline{u}$ sul bordo.
- C. al bordo si dicono NATURALI se cons. nell'assegnaz. del fattore di variazione dell'int. di bordo
  Qui sono l'assegn. di $\hat{\underline{t}}$.

C. ess. contribuiscono a definire l'INSIEME in cui va cercata la soluzione
C. naturali contr. alla forma delle eq. variaz.

L'eq. var. si può ottenere def. il LAVORO VIRTUALE ESTERNO:

$$\boxed{L_{ve} = \int_C \underline{b} \cdot \underline{v} + \int_{\partial C_2} \hat{\underline{t}} \cdot \underline{v}}$$ con $\underline{v}$: spost. virtuale

Impost. che $\underline{v} = 0$ su ∂C_1 (stessa prop. della variazione, è variaz. con norme di verso). [Ricordo: $L_{vi} = \int_C \underline{T} \cdot \nabla \underline{v}$]
Suppon. corpo in equil. Allora $\boxed{L_{ve} = L_{vi}}$ $\forall \underline{v}$

DIM:

$$L_{ve} = \int_C \operatorname{div} \underline{T} \cdot \underline{v} + \int_{\partial C_2} \hat{\underline{f}} \cdot \underline{v} = -\int_C \left(\operatorname{div} \underline{T}^T \underline{v} - \underline{T} \cdot \nabla \underline{v} \right) +$$

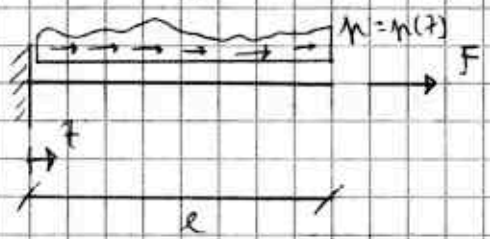
$$+ \int_{\partial C_2} \hat{\underline{f}} \cdot \underline{v}$$ Applico th. diver.:

$$L_{ve} = -\int_{\partial C_2} \underline{T}\underline{n} \cdot \underline{v} + \int_C \underline{T} \cdot \nabla \underline{v} + \int_{\partial C_2} \hat{\underline{f}} \cdot \underline{v} = -\int_C \left(\operatorname{div} \underline{T}^T \underline{v} - \underline{T} \cdot \nabla \underline{v} \right) +$$

$$+ \int_{\partial C_2} \hat{\underline{f}} \cdot \underline{v} = \int_{\partial C_2} \left(\hat{\underline{f}} - \cancel{\underline{T}\underline{n}} \right) \cdot \underline{v} + \int_C \underline{T} \cdot \nabla \underline{v} \quad \text{CVD}$$

È vero anche il contrario, infatti $L_{ve} = L_{vi}$ è proprio l'eq. variaz.

Form. variaz. possibile $\forall$ eq. diff. PLV e interpret. fisica legata all'equil. dei corpi.



Ex. applic. a th. travi.

Sez. costante e omog.: EA cost. con z.

$$\boxed{EA\, w'' + p = 0}, \text{ in } (0, l)$$

$w(0) = 0$, $EA\, w'(l) = F$ ↳ Prob. in forma classica.

Eq. variaz.: $\eta(0) = 0$

$$\int_0^l (EA\, w'' + p)\, \eta \, dz = 0 = \int_0^l \left((EA\, w' \eta)' - EA\, w' \eta' + \right.$$

$$\left. + p\, \eta \right) dz = 0 = \left[EA\, w' \eta \right]_0^l - \text{"} + \text{"} =$$

$$EA\, w'(l)\, \eta(l) - \cancel{EA\, w'(0)\, \eta(0)} - \int_0^l (EA\, w' \eta' - p\, \eta)\, dz$$

$$\boxed{\int_0^l (EA\, w' \eta' - p\, \eta)\, dz - \underbrace{EA\, w'(l)}_{F}\, \eta(l) = 0}$$

la sol. dell'eq. è la w tale che $w(0)=0$ e che la soddisfa $\forall\, \eta \;/\; \eta(0)=0$.
Supponendo valide le prop. di reg., dim. che è sol. anche per prob. classico. DIM.

$$\int_0^l (EJ w'\eta)' - \int_0^l (EJ w''\eta + p\,\eta)\,dz - F\eta(l) = 0$$

$$\Big[EJ w'\eta\Big]_0^l \quad EJ w'(l)\eta(l) - \;''\; - \;''\; = 0$$

$$EJ w'(l)\eta(l) - \cancel{EJ w'(0)\eta(0)}$$

$$= -\int_0^l (EJ w'' + p)\,\eta\,dz + (EJ w'(l) - F)\eta(l) = 0$$

da soddisfare $\forall\, \eta$.
Prendiamo $\eta(l)=0$ il 1° termine $=0$, allora x stessa dim. di prima se $I(x)$ è arbitrario $=0$, quindi $EJ w'' + p = 0$ in $(0,l)$. Quindi il 1° t. $=0$ e allora $EJ w'(l) = F$
Allora è sol. del prob. class. CVS.

Ex:

- $EJ\, v^{IV} - q = 0$ in $(0,l)$
  $v(0) = 0, \quad -v'(l) = 0$
- $-EJ\, v''(0) = -m$
- $-EJ\, v'''(l) = F$. Formule classica.

e Form. variaz.:

$$\int_0^l (EJ v^{IV} - q)\,\eta = 0 \quad \text{con } \eta(0)=0 \;;\; \eta'(l)=0$$

(54) Consr. $EJ v^{IV}\eta = (EJ v'''\eta) - EJ v'''\eta' =$

$(EJ v''' \eta)' = (EJ v'' \eta')' + EJ v'' \eta''$ (a si fanno quando sappiamo stime di derivat.)

Sost.: $0 = \left[EJ v''' \eta\right]_0^l - \left[EJ v'' \eta''\right]_0^l + \int_0^l (EJ v'' \eta'' + - q\eta)\, dz$.

Sfruttiamo c. al contorno. Si ha:

$$EJ v'''(l)\eta(l) + EJ v''(0)\eta'(0) + \quad // \quad =$$

$$\boxed{-F\eta(l) + m\eta'(0) + \int_0^l (EJ v'' \eta'' - q\eta)\, dz = 0}$$

Problema è det. v / $v(0)=0$, $v'(l)=0$ e soddisfa eq. $\forall \eta$ / $\eta(0)=0$, $\eta'(l)=0$.

$EJ v'' \eta'' = (EJ v'' \eta')' - EJ v'' \eta' = (EJ v'' \eta')' + - (EJ v''' \eta)' + EJ v^{IV} \eta$. Allora

$$\int_0^l (EJ v^{IV} - q)\eta\, dz - EJ v''(0)\eta'(0) - EJ v''(l)\eta(l) - F\eta(l) + m\eta'(0) = 0$$

$$\int_0^l (EJ v^{IV} - q)\eta\, dz + (m - EJ v''(0))\eta(0) + - (EJ v'''(l) + F)\eta(l) = 0.$$

Tutti i fattori di η o di η' sono singolarmente $=0 \Rightarrow$ è sol anche del p. Clam.

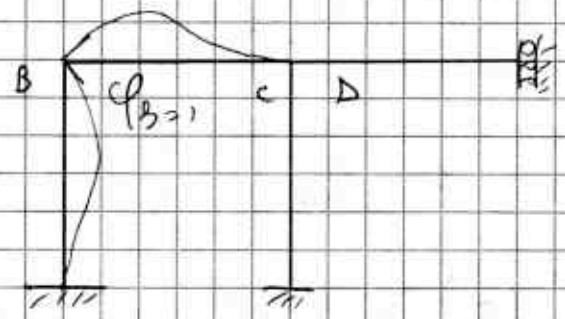
24/3/09

Problema di equilibrio

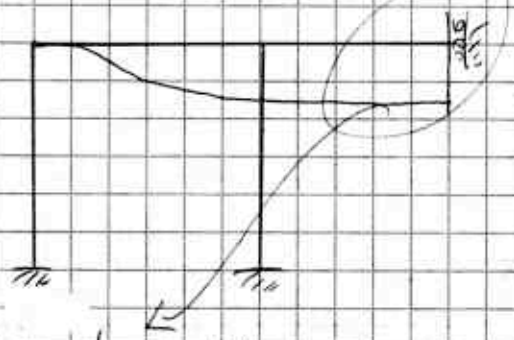
$$\begin{cases} \operatorname{div} \mathbb{C}[\nabla \underline{u}] + \underline{b} = \underline{0} & \text{in } C \\ \underline{u} = \hat{\underline{u}} & \text{su } \partial C_1, \quad \mathbb{C}[\nabla \underline{u}]\underline{n} = \hat{\underline{t}} \text{ su } \partial C_2 \end{cases}$$

$$\begin{cases} \int_C \mathbb{C}[\nabla \underline{u}] \cdot \nabla \underline{v} - \int_C \underline{b} \cdot \underline{v} - \int_{\partial C_2} \hat{\underline{t}} \cdot \underline{v} = 0 \quad , \forall \underline{v} \\ \underline{u} = \hat{\underline{u}} \text{ su } \partial C_1 \quad , \quad \underline{v} = \underline{0} \text{ su } \partial C_1 \end{cases}$$

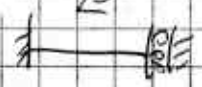
Sta:



$$\begin{bmatrix} v_B \\ w_B \\ \varphi_B \\ v_C \\ w_C \\ \varphi_C \end{bmatrix}$$



Si può cons. reciprocità dell'opera



Elastico: Corpo in cui $\underline{T} = \underline{T}(\nabla \underline{u})$, dip. dal valore att.

Lin. elastico: $\underline{T} = \mathbb{C}[\nabla \underline{u}]$, $\mathbb{C}[\underline{E}] = \mathbb{C}[\nabla \underline{u}]$, $\mathbb{C}[\underline{W}] = 0 \ \forall \underline{W} \in \mathrm{Skw}$

IPERELASTICO: $\exists\, \sigma \,/\, \underline{T} = \dfrac{\partial \sigma}{\partial \underline{E}} = D\sigma(\underline{E})$

$\sigma : \underline{E} \in \mathrm{Sym} \longmapsto \sigma(\underline{E}) \in \mathbb{R}$

$$\sigma(\underline{E} + \underline{S}) - \sigma(\underline{E}) = D\sigma(\underline{E})[\underline{S}] + o(\underline{S})$$

Per il th. di rapp. , $D\sigma(\underline{E})[\underline{S}] = \underbrace{D\sigma(\underline{E})}_{\underline{T}} \cdot \underline{S}$

Forma di σ per mat. Iperel.

Componenti: $T_{ij} = \underline{T} \cdot (\underline{e}_i \otimes \underline{e}_j) = \dfrac{\partial \sigma}{\partial \underline{E}} \cdot (\underline{e}_i \otimes \underline{e}_j) =$

$= D\sigma(\underline{E})[\underline{e}_i \otimes \underline{e}_j] = \dfrac{d}{d\lambda} \sigma(\underline{E} + \lambda(\underline{e}_i \otimes \underline{e}_j)) \Big|_{\lambda=0} =$

$= \dfrac{\partial \sigma}{\partial E_{ij}}$. Pari a $\lim\limits_{\lambda \to 0} \dfrac{\sigma(E_{11} + \lambda, E_{12} \dots E_{33}) - \sigma(E_{11}, E_{12} \dots E_{33})}{\lambda} =$

$= \dfrac{\partial \sigma}{\partial E_{ij}}$. Quindi $\boxed{T_{ij} = \dfrac{\partial \sigma}{\partial E_{ij}}}$

$\underline{\xi}(t)$, $t \in (t_1, t_2)$ [deformazione nell'intervallo t_1, t_2]

$\underline{\xi}_1 = \underline{\xi}(t_1)$, $\underline{\xi}_2 = \underline{\xi}(t_2)$

- Si ha $dl = \underline{T} \cdot d\underline{\xi}$ [lavoro elementare] nello stesso punto ovviam.
- Si ha invece il lavoro per unità di volume $l = \int_{t_1}^{t_2} \underline{T} \cdot d\underline{\xi} =$

$$= \int_{t_1}^{t_2} \underline{T} \cdot \frac{d\underline{\xi}}{dt}\, dt = \int_{t_1}^{t_2} \frac{\partial \sigma}{\partial \underline{\xi}} \cdot \frac{d\underline{\xi}}{dt}\, dt = \int_{t_1}^{t_2} D\sigma(\underline{\xi}) \left[\frac{d\underline{\xi}(t)}{dt}\right] dt =$$

$$= \int_{t_1}^{t_2} \frac{d\sigma}{dt}\, dt = \sigma(t_2) - \sigma(t_1) = \sigma(\underline{\xi}_2) - \sigma(\underline{\xi}_1)$$

σ DENSITÀ DI ENERGIA ELASTICA

$$L = \int_C l\, dv = \int_C \sigma(\underline{\xi}_2)\, dv - \int_C \sigma(\underline{\xi}_1)\, dv$$

$$l = \sigma(\underline{\xi}_2) - \sigma(\underline{\xi}_1)$$

Con percorso di deform. chiuso. Allora $l = 0$

$\underline{\xi}(t) = \underline{S}_1 \cos t + \underline{S}_2 \sin t$; $t \in (0, 2\pi)$, $\underline{S}_1, \underline{S}_2 \in Sym$

$\underline{T} = \mathbb{C}[\underline{\xi}] = \mathbb{C}[\underline{S}_1] \cos t + \mathbb{C}[\underline{S}_2] \sin t$

$\frac{d\underline{\xi}}{dt} = -\underline{S}_1 \sin t + \underline{S}_2 \cos t$

$\underline{T} \cdot \frac{d\underline{\xi}}{dt} = \left(\mathbb{C}[\underline{S}_1] \cos t + \mathbb{C}[\underline{S}_2] \sin t\right) \cdot \left(-\underline{S}_1 \sin t + \underline{S}_2 \cos t\right) =$

$= -\sin t \cos t\, \mathbb{C}[\underline{S}_1] \cdot \underline{S}_1 + \cos^2 t\, \mathbb{C}[\underline{S}_1] \cdot \underline{S}_2 - \sin^2 t\, \mathbb{C}[\underline{S}_2] \cdot \underline{S}_1 +$
$+ \sin t \cos t\, \mathbb{C}[\underline{S}_2] \cdot \underline{S}_2$;

$l = \int_0^{2\pi} \underline{T} \cdot \frac{d\underline{\xi}}{dt}\, dt$ $\left[\int_0^{2\pi} \sin \cos = 0\right.$; $\cos 2t = \cos^2 t - \sin^2 t =$

$= 1 - 2\sin^2 t = 2\cos^2 t - 1$; $\cos^2 t = \frac{1}{2}(\cos 2t + 1)$; $\sin^2 t =$
$= \frac{1}{2}(1 - \cos 2t)$].

Integrando in sostanza si ha $\ell = \tilde{\eta}\left(\mathbb{C}[\underline{S}_1]\cdot\underline{S}_2 - \mathbb{C}[\underline{S}_2]\cdot\underline{S}_1\right) = 0$

Allora $\mathbb{C}[\underline{S}_1]\cdot\underline{S}_2 = \mathbb{C}[\underline{S}_2]\cdot\underline{S}_1 \Rightarrow$ <u>$\mathbb{C}$ e' sym.</u>

sym e' conseguenza naturale dell'iperelasticita'.

Se $\exists\ \sigma$, $\mathbb{C} \in$ sym

Def. $\boxed{\sigma(\underline{\varepsilon}) = \frac{1}{2}\mathbb{C}[\underline{\varepsilon}]\cdot\underline{\varepsilon}}$; $\sigma = \frac{1}{2} C_{ijmn}\,\varepsilon_{ij}\,\varepsilon_{kl}$

Dim: $D\sigma(\underline{\varepsilon})[\underline{S}] = \frac{d}{d\lambda}\sigma(\underline{\varepsilon} + \lambda\underline{S})\Big|_{\lambda=0} =$

$= \frac{d}{d\lambda}\left[\frac{1}{2}\mathbb{C}[\underline{\varepsilon} + \lambda\underline{S}]\cdot(\underline{\varepsilon} + \lambda\underline{S})\right]_{\lambda=0} = \left[\frac{1}{2}\mathbb{C}[\underline{S}]\cdot(\underline{\varepsilon} + \lambda\underline{S}) +\right.$

$\left.+\frac{1}{2}\mathbb{C}[\underline{\varepsilon} + \lambda\underline{S}]\cdot\underline{S}\right]_{\lambda=0} = \frac{1}{2}\mathbb{C}[\underline{S}]\,\underline{\varepsilon} + \frac{1}{2}\mathbb{C}[\underline{\varepsilon}]\cdot\underline{S} =$

(essendo $\mathbb{C} = \mathbb{C}^T$) $= \mathbb{C}[\underline{\varepsilon}]\cdot\underline{S} = \underline{T}\cdot\underline{S}$, ovvero $\frac{\partial\sigma}{\partial\underline{\varepsilon}} = \underline{T} \rightarrow$ Def. CONVERTITA C.V.D.

Se $\underline{\varepsilon}_1 = \underline{0}$, $\ell = \sigma(\underline{\varepsilon}_2) > 0$ $\forall\,\underline{\varepsilon}_2 \neq \underline{0}$ allora
$\mathbb{C}[\underline{\varepsilon}]\cdot\underline{\varepsilon} > 0$, $\forall\,\underline{\varepsilon} \neq 0 \Rightarrow$ <u>$\mathbb{C}$ e' def. positiva</u>

Corpo non si deforma se solo se $\underline{\varepsilon}_1 = \underline{0}$

$L = \int_C \sigma(\underline{\varepsilon}_2) - \int_C \sigma(\underline{\varepsilon}_1)$; se $\underline{\varepsilon}_1 = 0$, $L = \int_C \sigma(\underline{\varepsilon}) =$

$= \frac{1}{2}\int_C \underline{T}\cdot\underline{\varepsilon} = \frac{1}{2}\int_C \underline{T}\cdot\nabla\underline{u} =$ [sappiamo che $\operatorname{div}\underline{T}\cdot\underline{u} +$

$\underline{T}\cdot\nabla\underline{u} = \operatorname{div}(\underline{T}^T\underline{u})$] $= \frac{1}{2}\int_C\left(\operatorname{div}\underline{T}^T\underline{u} - \operatorname{div}\underline{T}\cdot\underline{u}\right) =$

$= \frac{1}{2}\int_{\partial C}\underline{T}\underline{n}\cdot\underline{u} + \frac{1}{2}\int_C \underline{b}\cdot\underline{u}$

58 Allora $\boxed{L = \frac{1}{2}\int_C \underline{b}\cdot\underline{u} + \frac{1}{2}\int_{\partial C}\underline{t}\cdot\underline{u}} \rightarrow$ TH. DI CLAPEYRON

FUNZIONALE DELL'ENERGIA associato a prob. di equil.

$$\boxed{\Pi(\underline{u}) = \int_C \sigma \, dv - \int_C \underline{b} \cdot \underline{u} - \int_{\partial C_2} \hat{\underline{t}} \cdot \underline{u}}$$

Vogliamo dim. che ha min $\equiv$ sol. prob. di equil.

Funzionale: $f(\mathbb{R}) \longmapsto \mathbb{R}$

$$\Pi(\underline{u}) = \underbrace{\frac{1}{2}\int_C \mathbb{C}[\nabla \underline{u}] \cdot \nabla \underline{u}}_{\frac{1}{2}\int_C \underline{T} \cdot \underline{E}} - \int_C \underline{b} \cdot \underline{u} - \int_{\partial C_2} \hat{\underline{t}} \cdot \underline{u}$$

Supponiamo $\tilde{\underline{u}}$ soluzione e quindi $\min \Pi \equiv \tilde{\underline{u}}$.

Conv. $\Pi(\underline{u}) - \Pi(\tilde{\underline{u}}) =$

$$\mathcal{U} = \left\{ \underline{u} : \underline{u} = \hat{\underline{u}} \text{ su } \partial C_1, \ \underline{u} \in C^2(C), \ \underline{u} \in C^1(\partial C) \right\}$$

↳ Insieme delle fun. CINEMATICHE AMMISS.

$\Pi : \mathcal{U} \longmapsto \mathbb{R}$

$$= \frac{1}{2}\int_C (\underline{T} \cdot \underline{E} - \tilde{\underline{T}} \cdot \tilde{\underline{E}}) - \int_C \underline{b} \cdot (\underline{u} - \tilde{\underline{u}}) - \int_{\partial C_2} \hat{\underline{t}} \cdot (\underline{u} - \tilde{\underline{u}})$$

Conv. 3° membro: $-\int_{\partial C_2} \hat{\underline{t}} \cdot (\underline{u} - \tilde{\underline{u}}) = -\int_{\partial C_2} \tilde{\underline{T}} \, \underline{n} \cdot (\underline{u} - \tilde{\underline{u}}) =$

$$= -\int_{\partial C} \tilde{\underline{T}}^T (\underline{u} - \tilde{\underline{u}}) \cdot \underline{n} = -\int_C \operatorname{div}\left(\tilde{\underline{T}}^T (\underline{u} - \tilde{\underline{u}})\right) dv =$$

$$= -\int_C \left(\operatorname{div} \tilde{\underline{T}} \cdot (\underline{u} - \tilde{\underline{u}}) + \tilde{\underline{T}} \cdot \nabla(\underline{u} - \tilde{\underline{u}})\right) = \int_C (-\underline{b}) \cdot (\underline{u} - \tilde{\underline{u}}) +$$

$$- \int_C \tilde{\underline{T}} \cdot (\underline{E} - \tilde{\underline{E}})$$ Torna alla suff:

$$= \frac{1}{2}\int_C (\underline{T} \cdot \underline{E} - \tilde{\underline{T}} \cdot \tilde{\underline{E}}) - \int_C \tilde{\underline{T}} \cdot (\underline{E} - \tilde{\underline{E}}) = \frac{1}{2}\int_C \left(\underline{T} \cdot \underline{E} - \cancel{\tilde{\underline{T}} \cdot \tilde{\underline{E}}} + \right.$$

$$- 2\underbrace{\tilde{\underline{T}} \cdot \underline{\xi}} + \cancel{2}\tilde{\underline{T}} \cdot \tilde{\underline{\xi}} \Big) \qquad \text{Allora:}$$

$$2\,\mathbb{C}[\tilde{\underline{\xi}}] \cdot \underline{\xi} = \mathbb{C}[\tilde{\underline{\xi}}] \cdot \underline{\xi} + \tilde{\underline{\xi}} \cdot \mathbb{C}[\underline{\xi}]$$

$$\tilde{\Pi}(\underline{u}) - \tilde{\Pi}(\tilde{\underline{u}}) = \frac{1}{2}\int_C (\underline{T} - \tilde{\underline{T}}) \cdot (\underline{\xi} - \tilde{\underline{\xi}}) = \frac{1}{2}\int_C \mathbb{C}[\underline{\xi} - \tilde{\underline{\xi}}] \cdot (\underline{\xi} - \tilde{\underline{\xi}})$$

≥ 0. Per essere nullo, $\underline{\xi} = \tilde{\underline{\xi}}$ altrimenti > 0 con $\underline{\xi} \neq \tilde{\underline{\xi}}$. C.V.D.

$\tilde{\Pi}(u) > \tilde{\Pi}(\tilde{u}) \;\forall\, \underline{\xi} \neq \tilde{\underline{\xi}}$ differiscono per spost. rigido

Def insieme variazioni in u.

$$\delta u = \left\{ \delta \underline{u} : \delta \underline{u} = \underline{0} \text{ su } \partial C_1,\ \delta \underline{u} \in C^2(C),\ \delta \underline{u} \in C^1(\partial C) \right\}$$

Conv. $\underline{u} + \lambda\, \delta \underline{u}$, $[\underline{u} \in \mathcal{U},\ \delta \underline{u} \in \delta u] \in \mathcal{U}$

Allora poniamo $\tilde{\Pi}(\underline{u} + \lambda\, \delta \underline{u})$. Se scegliamo $\tilde{\underline{u}}$ ho $\tilde{\Pi}$ che per $\lambda = 0$ ha min. $\left.\frac{d\tilde{\Pi}}{d\lambda}\right|_{\lambda=0} = 0$.

$\delta \tilde{\Pi} = \frac{d\tilde{\Pi}}{d\lambda}$ detta VARIAZIONE di $\tilde{\Pi}$.

Meglio $\underline{\delta \tilde{\Pi}(\underline{u})\{\delta \underline{u}\}}$. Sviluppo: (IP: $\delta\tilde{\Pi} = 0 \;\forall\, \delta \underline{u} \in \delta u$)

$$\delta \tilde{\Pi} = \frac{d}{d\lambda} \tilde{\Pi}(\underline{u} + \lambda\, \delta \underline{u})\Big|_{\lambda=0} = \frac{d}{d\lambda}\Bigg[\frac{1}{2}\int_C \mathbb{C}[\nabla \underline{u} + \lambda \nabla \delta \underline{u}] \cdot$$

$$(\nabla \underline{u} + \lambda \nabla \delta \underline{u}) - \int_C \underline{b} \cdot (\underline{u} + \lambda\, \delta \underline{u}) - \int_{\partial C_2} \hat{\underline{t}} \cdot (\underline{u} + \lambda\, \delta \underline{u})\Bigg]_{\lambda=0} =$$

$$= \int_C \mathbb{C}[\nabla \underline{u}] \cdot \nabla \delta \underline{u} - \int_C \underline{b} \cdot \delta \underline{u} - \int_{\partial C_2} \hat{\underline{t}} \cdot \delta \underline{u} =$$

$$= \int_C \left(\operatorname{div}(\underline{T}^T \delta \underline{u}) - \operatorname{div} \underline{T} \cdot \delta \underline{u} \right) + \Big[\mathbb{C}[\nabla \underline{u}] \cdot \nabla \delta \underline{u} = T \cdot \nabla \delta \underline{u} = \operatorname{div}(T^T \delta \underline{u}) - \operatorname{div} \underline{T} \cdot \delta \underline{u} \Big]$$

$$- \int_C \underline{b} \cdot \delta \underline{u} - \int_{\partial C_2} \hat{\underline{t}} \cdot \delta \underline{u} = - \int_C (\operatorname{div} \underline{T} + \underline{b}) \cdot \delta \underline{u} \quad +$$

$$\int_{\partial C_2} \underline{T}\,\underline{n} \cdot \delta \underline{u} - \int_{\partial C_2} \hat{\underline{t}} \cdot \delta \underline{u} = -\int_C (\operatorname{div} \underline{T} + \underline{b}) \cdot \delta \underline{u} + \int_{\partial C_2} (\underline{T}\,\underline{n} - \hat{\underline{t}}) \cdot \delta \underline{u}$$

valida $\forall\, \delta \underline{u}$. Essendo $\operatorname{div} \underline{T} + \underline{b} = \underline{0}$ in C e $\underline{T}\,\underline{n} = \hat{\underline{t}}$ su ∂C_2 la $\underline{u} + \lambda\, \delta \underline{u}$ è sol del prob. C.V.D.

$$\Downarrow$$

$$\boxed{\delta \Pi = 0 \quad , \quad \begin{array}{c} \forall\, \delta \underline{u} \in \delta \mathcal{U} \\ \underline{u} \in \mathcal{U} \end{array}}$$

30/3/09

[- Equil. in: forme diff. ①

$$\begin{cases} \operatorname{div} \mathbb{C}[\nabla \underline{u}] + \underline{b} = \underline{0} & \text{in } C \\ \underline{u} = \hat{\underline{u}} \text{ su } \partial C_1 \,;\quad \mathbb{C}[\nabla \underline{u}] = \hat{\underline{t}} & \text{su } \partial C_2 \end{cases}$$

②- Forme variaz.:

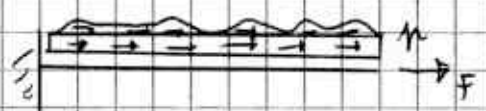
$$\int_C \underline{T} \cdot \nabla \underline{v}\, dv - \int_C \underline{b} \cdot \underline{v}\, dv - \int_{\partial C} \hat{\underline{t}} \cdot \underline{v} = 0 \quad , \quad \begin{array}{l} \underline{u} = \hat{\underline{u}} \text{ su } \partial C_1 \\ \forall\, \underline{v} \,,\ \underline{v} = \underline{0} \text{ su } \partial C_1 \end{array}$$

③- Funzionale spost.:

$$\Pi(\underline{u}) = \int_C \sigma(\underline{\varepsilon}(\underline{u}))\, dv - \int_C \underline{b} \cdot \underline{u} - \int_{\partial C_2} \hat{\underline{t}} \cdot \underline{u} \quad (\min \equiv \text{sol.})$$

$$\delta \Pi(\underline{u}) = \frac{d}{d\lambda} \Pi(\underline{u} + \lambda \underline{u}) \Big|_{\lambda = 0} = 0 \,,\ \forall\, \delta \underline{u} \,,\ \delta \underline{u} = \underline{0} \text{ su } \partial C_1 \,]$$

Ex:



$$\sigma = \frac{1}{2} \underline{T} \cdot \underline{\varepsilon} = \frac{1}{2} (T_{11} \varepsilon_{11} + T_{22} \varepsilon_{22} + T_{33} \varepsilon_{33})$$

Qui abbiamo solo σ_z (nella generale abbiamo 3 ter. invece)

$w(0) = 0$

Quindi si ha $\sigma = \frac{1}{2} \sigma_z \varepsilon_z = E \varepsilon_z = \frac{1}{2E} \sigma_z^2 = \frac{1}{2} E \varepsilon_z^2$

con $\varepsilon_z = \frac{\partial w}{\partial z} = w'$ quindi $\sigma = \frac{1}{2} E (w')^2$

Cons. su tutto il corpo:

$$\int_C \sigma\, dv = \frac{1}{2} \int_0^l \int_A E (w')^2 \Rightarrow \Pi(w) = \frac{1}{2} \int_0^l E A (w')^2\, dz \quad +$$

$-\int_0^l p\, w\, dz - F w(l)$. W deve essere cinematic. ammissibile, quindi qui

$w \in W = \{ w / w \in C^2(0,l),\ w(0) = 0 \}$

$\delta w \in \delta W = \{ \delta w / \delta w \in C^2(0,l),\ \delta w(0) = 0 \}$

Allora $(w + \lambda \delta w) \in W$ → ha senso calc. il funz

$$\frac{d}{d\lambda} \Pi(w + \lambda w)\Big|_{\lambda=0} = \frac{d}{d\lambda}\left[\frac{1}{2}\int_0^l EJ(w' + \lambda \delta w')^2 dz + \int_0^l p(w + \lambda \delta w) - F(w(l) + \lambda \delta w(l)) \right]_{\lambda=0}$$

$$\delta \Pi = \int_0^l EA\, w' \delta w' dz - \int_0^l p\, \delta w - F \delta w(l) =$$

$$= \int_0^l \left((EA(w' \delta w))' - EJ w'' \delta w \right) dz - \int_0^l p\, \delta w - F \delta w(l) =$$

$$= \Big[EA\, w' \delta w \Big]_0^l - \int_0^l (EJ w'' + p) \delta w - F \delta w(l) =$$

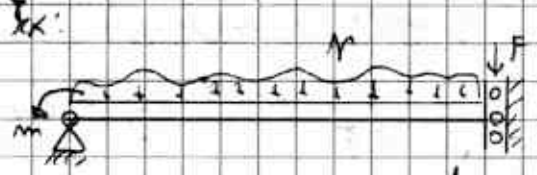
$$= -\int_0^l (EJ w'' + p) \delta w + (EJ w'(l) - F) \delta w(l) = 0$$

Valido $\forall \delta w$, quindi (ritrovate le eq. di equil. in forma classica)

c.v.d.

$$\boxed{\begin{array}{ll} EJ w'' + p = 0 & \text{in } (0,l) \\ EJ w'(l) = F & \end{array}}$$

Es:



$v(0) = 0 \qquad -v'(l) = 0$

$$\sigma = \frac{1}{2}\sigma_z \varepsilon_z = \frac{1}{2}\frac{1}{E}\sigma_z^2 = \frac{1}{2E}\left(\frac{M_x}{J_x} y\right)^2 = \frac{1}{2} E (v'')^2 y^2.$$

Integrando su A, $\Pi(v) = \frac{1}{2}\int_0^l EJ (v'')^2 - \int_0^l q v - m(-v'(0)) - F v(l)$

$$\delta\tilde{\Pi} = \frac{d}{d\lambda}\tilde{\Pi}(v+\lambda\delta v)\Big|_{\lambda=0} = \frac{d}{d\lambda}\Big[\frac{1}{2}\int_0^l EJ(v''+\lambda\delta v'')^2 +$$

$$-\int_0^l q(v+\lambda\delta v) + m(v'(0)+\lambda\delta v'(0)) - F(v(l)+\lambda\delta v(l))\Big]_{\lambda=0} =$$

$$= \int_0^l EJ v''\delta v'' - \int_0^l q\,\delta v + m\,\delta v'(0) - F\,\delta v(l)$$

$$\Big[\ EJ v''\delta v' = (EJ v''\delta v')' - EJ v'''\delta v' = (EJ v''\delta v')' +$$

$$-(EJ v'''\delta v)' + EJ v^{IV}\delta v\ \Big]$$

$$\delta\tilde{\Pi} = \int_0^l (EJ v^{IV} - q)\,\delta v + \Big[EJ v''\delta v'\Big]_0^l - \Big[EJ v'''\delta v\Big]_0^l + m\,\delta v'(0) - F\,\delta v(l) =$$

$$\delta\tilde{\Pi} = \int_0^l (EJ v^{IV} - q)\,\delta v - (EJ v''(0) - m)\,\delta v'(0) - (EJ v'''(l) + F)\,\delta v(l) = 0$$

allora $\boxed{\begin{array}{ll} EJ v^{IV} = q & -EJ v''(0) = -m \\ \text{in } (0,l) & -EJ v'''(l) = F \end{array}}$ C.V.D.

Metodi introdotti x giust. ricerca risultati F. ottimi x sol. approx. [SOLUZIONI APPROSSIMATE DEL PROBLEMA D'EQUILIBRIO]

- Cons. funz. energia prob. equil. corpo continuo:

$$\tilde{\Pi} = \frac{1}{2}\int_C \mathbb{C}[\nabla \underline{u}]\cdot\nabla\underline{u} - \int_C \underline{b}\cdot\underline{u} - \int_{\partial C_2} \hat{\underline{t}}\cdot\underline{u}$$

Vogliamo sol. approx. del min.

Poniamo $\boxed{\underline{u}(x) = \underline{\phi}_0(x) + \sum_{i=1}^{m} \alpha_i\,\underline{\phi}_i(x)}$

Vettore funz. del punto — coeff.

Poniamo $\underline{\phi}_0 = \hat{\underline{u}}$ su ∂C_1, $\underline{\phi}_i = \underline{0}$ su ∂C_1 per rispett. la cond. essenz.

Sostituiamo: (le ϕ fissate le α_i incognite):

$$\tilde{\Pi} = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\underbrace{\int_C \mathbb{C}[\nabla\underline{\phi}_i]\cdot\nabla\underline{\phi}_j}_{k_{ij}}\,\alpha_i\alpha_j + \Big(\sum_{j=1}^{m}\int_C \mathbb{C}[\nabla\underline{\phi}_0]\cdot\nabla\underline{\phi}_j +$$

$$\underbrace{-\int_C \underline{b}\cdot\underline{\phi}_J - \int_{\partial C_2} \hat{\underline{t}}\cdot\underline{\phi}_J}_{q_J}\Big)\alpha_J + \underbrace{\frac{1}{2}\int_C \mathbb{C}[\nabla\underline{\phi}_0]\cdot\nabla\underline{\phi}_0 - \int_C \underline{b}\cdot\underline{\phi}_0 - \int_{\partial C_2}\hat{\underline{t}}\cdot\underline{\phi}_0}_{s}$$

$$\tilde{\Pi}(\alpha_1 \dots \alpha_n) = \frac{1}{2}\sum_{i=1}^{n}\sum_{J=1}^{n} K_{iJ}\,\alpha_i\,\alpha_J + \sum_{J=1}^{n} q_J\,\alpha_J + s$$

$$\frac{\partial\tilde{\Pi}}{\partial\alpha_k} = \sum_{J=1}^{n} K_{kJ}\,\alpha_J + q_k = 0$$ Sist. di n eq. nelle incognite $\alpha_1 \dots \alpha_n$.

imponiamo $= 0$ per $k = 1 \dots n$

$$\boxed{\underline{u}(x) = \underline{\phi}_0 + \sum_{i=1}^{n}\alpha_i\,\underline{\phi}_i}$$ METODO DI RITZ

$$-\int_C \underline{T}\cdot\nabla\underline{v} - \int_C \underline{b}\cdot\underline{v} - \int_{\partial C_2}\hat{\underline{t}}\cdot\underline{v} = 0$$

$\underline{u} = \hat{\underline{u}}$ su ∂C_1

$\underline{v} = \underline{0}$ su ∂C_1

$\underline{v} = \sum_{J=1}^{n}\beta_J\,\underline{\phi}_J$ (stesse ϕ di prima ma altri coeff.)

$$\sum_{i=1}^{n}\sum_{J=1}^{n}\Big(\int_C \mathbb{C}[\nabla\underline{\phi}_i]\cdot\nabla\underline{\phi}_J\Big)\alpha_i\beta_J + \sum_{J=1}^{n}\Big[\int_C \mathbb{C}[\nabla\underline{\phi}_0]\cdot\nabla\underline{\phi}_J +$$

$$-\int_C \underline{b}\cdot\underline{\phi}_J - \int_{\partial C_2}\hat{\underline{t}}\cdot\underline{\phi}_J\Big]\beta_J = 0$$

$$\sum_{i=1}^{n}\sum_{J=1}^{n} K_{iJ}\,\alpha_i\,\beta_J + \sum_{J=1}^{n} q_J\,\beta_J = 0$$ β_J arbitrari.

Si può scegliere $\beta_1 = 1$ e $\beta_2 = \dots = \beta_n = 0$

$\beta_2 = 1$, e $\beta_1 = \beta_3 = \dots = \beta_n = 0$

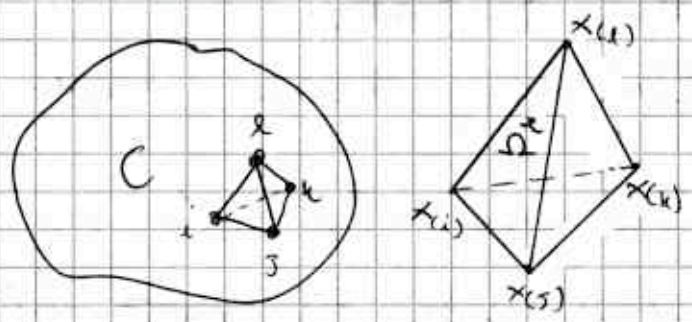
(cioè n scelte con 1 unitario e altri nulli, a scalare)

(64) $$\boxed{\sum_{J=1}^{n} K_{Ji}\,\alpha_i + q_J = 0}$$ con $J = 1, \dots, n$

METODO DI GALERKIN

Problema è scegliere le ϕ!

Aiuto dal FEM, costruisce in modo sistematico le ϕ.



Si suddivide corpo in parti di forma geometrica semplice, gli ELEMENTI FINITI. All'interno si scelgono i NODI.

Le ϕ_i si scelgono polinomiali, compatibili con il problema da studiare, definite $\forall$ elemento.

Ex: suddividiamo corpo in tetraedri con vertici i, j, k, l in x_i, x_j, x_k, x_l scelti come nodi, mentre gli d sono i loro spostamenti

Definiamo le ϕ $\forall$ elem.

$$\underline{u}^e(x) = \begin{bmatrix} u_1^e(x) \\ u_2^e(x) \\ u_3^e(x) \end{bmatrix}$$; gli spostamenti sono $\underline{S}_{(i)} = \begin{bmatrix} S_{(i)1} \\ S_{(i)2} \\ S_{(i)3} \end{bmatrix}$

Conv. vett. spost $\underline{S}^e = \begin{bmatrix} \underline{S}_{(i)} \\ \underline{S}_{(j)} \\ \underline{S}_{(k)} \\ \underline{S}_{(l)} \end{bmatrix}$. Possiamo scrivere:

$$\underline{u}^e(x) = \underline{\Phi}^e \, \underline{S}^e$$

$$u_1^e(x) = \phi_i^e(x) S_{(i)1} + \phi_j^e(x) S_{(j)1} + \phi_k^e(x) S_{(k)1} + \phi_l^e(x) S_{(l)1}$$

In not. matr. $$\underline{u}^e(x) = \begin{bmatrix} \phi_i^e & 0 & 0 & \phi_j^e & 0 & 0 & \phi_k^e & 0 & 0 & \phi_l^e & 0 & 0 \\ 0 & \psi_i^e & 0 & 0 & \psi_j^e & 0 & 0 & \psi_k^e & 0 & 0 & \psi_l^e & 0 \\ 0 & 0 & \chi_i^e & 0 & 0 & \chi_j^e & 0 & 0 & \chi_k^e & 0 & 0 & \chi_l^e \end{bmatrix} \cdot$$

$$\begin{bmatrix} S_{(i)1} & S_{(i)2} & S_{(i)3} & S_{(j)1} & \dots & S_{(l)3} \end{bmatrix}^T$$

Nella form. variaz. abbiamo solo der. prime, quindi ci basta pol. $\in C^1$. $\rightarrow b = i, j, k, l$

$$\phi_b^e(x_1, x_2, x_3) = a_{b0} + a_{b1} x_1 + a_{b2} x_2 + a_{b3} x_3$$

$\psi_b^e(x_1, x_2, x_3) = \bar{a}_{b0} + \bar{a}_{b1} x_1 + \bar{a}_{b2} x_2 + \bar{a}_{b3} x_3$, $b = i, j, k, l$

χ_b^e ... Sono dette FUNZIONI DI FORMA.

C'è rel. tra grado polinom. e num. nodi.

$u_1^e(x) = \phi_i^e S_{(i)1} + \phi_j^e S_{(j)1} + \phi_k^e S_{(k)1} + \phi_l^e S_{(l)1}$

Coeff. polinomi derivano dal fatto che coeff. delle corrm. lin. siano gli spost. nodali. Allora:

$\phi_b^e(x_{(c)}) = \delta_{bc}$ con $b, c = i, j, k, l$

Idem per le $\psi_b^e(x_{(c)}) = \delta_{bc}$ e $\chi_b^e(x_{(c)}) = \delta_{bc}$

Ho num. condizioni = num. coeff.!

31/3/2009

Metodo elementi finiti:

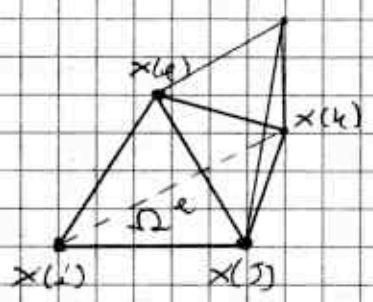
1) Si suddivide il corpo in sottoparti dalla forma semplice detti "elem. finiti" e si individuano in ciascun elemento dei punti particolari, i "nodi"

2) Si esprime ciascuna componente di spostamento (generalizzate, anche rotazioni ad ex.) nella forma $S_i = \sum_{k=1}^{m} \lambda_k^{(i)} \phi_k^{(i)}$, le ϕ_k sono definite elemento per elemento (ϕ_k^e è la restrizione di ϕ_k all'elemento "e") e hanno forma semplice (polinomi)

3) Si utilizzano le espress. appross. di S_i col metodo di Ritz o col metodo di Galerkin oppure si considera l'equilibrio di 1 elemento e si costruisce un sistema di equazioni imponendo condizioni di equilibrio nodale (metodo degli spostamenti)

4) Si impongono le cond. essenziali, si riducono

le forze esterne a carichi nodali si risolve il sistema determinando i parametri di spostam. nodali.



$$\underline{u}^e(x) = \underline{\Phi}^e(x)\, \underline{s}^e$$

Conv. altro tetraedro affiancato che condivide 3 nodi.

$$\underline{u}^e(x) = \begin{bmatrix} u_1^e(x) \\ u_2^e(x) \\ u_3^e(x) \end{bmatrix} \quad u_1^e(x) = s_{(i),1}\, \phi_i^e + s_{(j),1}\, \phi_j^e + s_{(k),1}\, \phi_k^e + s_{(l),1}\, \phi_l^e$$

$$\phi_a^e(x_{(b)}) = \delta_{ab} \quad ; \quad \phi_i^e = a_{i0} + a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3$$

$$\phi_i^e(x_{(i)}) = 1 \; ; \; \phi_i^e(x_{(j)}) = 0 \; ; \; \phi_i^e(x_{(k)}) = 0 \; , \; \phi_i^e(x_{(l)}) = 0$$

Il tetr. non cont. nodo i quindi è ha 4 cond. nulle $[\phi_i^e = 0 \;\forall$ nodo$]$

Si può scrivere quindi $u_1(x) = \sum_{m=1}^{N} s_{(m),1}\, \phi_m$

Le variazioni $v_1(x) = \sum_{m=1}^{N} \beta_{(m),1}\, \phi_m$

È + semplice conv. equilibrio singolo elem. e costruire equaz.

Abbiamo $\underline{u}^e = \underline{\Phi}(x)\, \underline{s}^e$ e associamo una misura di deformazione $\underline{\varepsilon}^e = \underline{D}\, \underline{u}^e = \underline{D}\, \underline{\Phi}^e\, \underline{s}^e$ ovvero:

$= [\varepsilon_{11}^e, \varepsilon_{22}^e, \varepsilon_{33}^e, 2\varepsilon_{12}^e, 2\varepsilon_{23}, 2\varepsilon_{31}]$ (per scorrimenti) =

$$\underline{\varepsilon}^e = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{bmatrix}}_{\to \underline{D}} \begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{bmatrix}$$

Si può scrivere $\underline{\varepsilon}^e = \underline{B}^e\, \underline{s}^e$

Introduciamo una misura della tensione:

$\underline{\sigma}^e = \{T_{11}, T_{22}, T_{33}, T_{12}, T_{23}, T_{31}\}$. Matrice 6×6

$\underline{\sigma}^e = \underline{C}\,\underline{\varepsilon}^e$. Se materiale isotropo:

$$\underline{T} = 2\mu\,\underline{E} + \lambda(\operatorname{tr}\underline{E})\,\underline{I}$$

$$\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

$\underline{\varepsilon}^e$

Allora $\underline{\sigma}^e = \underline{C}\,\underline{B}^e\,\underline{s}^e$

Quindi $\underline{u}^e = \underline{\Phi}^e\,\underline{s}^e$

$\underline{v}^e = \underline{\Phi}^e\,\underline{\beta}^e$ → arbitr., 12 comp.

$\underline{H}^e = \operatorname{sym}\nabla\underline{v}^e$ (variat.)

$$\{H_{11}^e, H_{22}^e, H_{33}^e, 2H_{12}^e, 2H_{23}^e, 2H_{31}^e\} = \underline{B}^e\underline{\beta}^e = \underline{D}\,\underline{v}^e = \underline{D}\,\underline{\Phi}^e\underline{\beta}^e$$

Passiamo sia Mecc. il prob. variat. in f. approx.

$$\int_{\Omega^e} \overset{\mathbb{C}[\nabla u]}{\underline{T}}\,\nabla\underline{v} - \int_{\Omega^e}\underline{b}\cdot\underline{v} - \int_{\partial\Omega^e}\underline{t}\cdot\underline{v} - \int_{\partial\Omega^e}\hat{\underline{t}}\cdot\underline{v} = 0$$

Sul contorno agiscono i carichi assegnati o forze da altri elementi. Se sono noti sono $\hat{t}$, se da altri elem sono incogniti. Sostituendo:

$$\int_{\Omega^e}\underline{\sigma}^e\cdot\underline{D}\,\underline{v}^e - \int_{\Omega^e}\underline{b}\cdot\underline{v}^e - \int_{\partial\Omega_e}\underline{t}\cdot\underline{v}^e - \int_{\partial\Omega^e}\hat{\underline{t}}\cdot\underline{v}^e = 0$$

$$\int_{\Omega^e}\underline{C}\,\underline{B}^e\,\underline{s}^e\cdot\underline{B}^e\,\underline{\beta}^e - \int_{\Omega^e}\underline{b}\cdot\underline{\Phi}^e\,\underline{\beta}^e - \int_{\partial\Omega^e}\underline{t}\cdot\underline{\Phi}^e\,\underline{\beta}^e - \int_{\partial\Omega^e}\hat{\underline{t}}\cdot\underline{\Phi}^e\,\underline{\beta}^e = 0$$

$$\int_{\Omega}\underline{B}^{e^T}\,\underline{C}\,\underline{B}^e\,\underline{s}^e\cdot\underline{\beta}^e - \int_{\Omega^e}\underline{\Phi}^{e^T}\underline{b}\cdot\underline{\beta}^e - \int_{\partial\Omega^e}\underline{\Phi}^{e^T}\underline{t}\cdot\underline{\beta}^e - \int_{\partial\Omega^e}\underline{\Phi}^{e^T}\hat{\underline{t}}\cdot\underline{\beta}^e = 0$$

$$0 = (\underline{k}^e\,\underline{s}^e + \underline{q}^e - \underline{r}^e)\cdot\underline{\beta}^e \quad \text{con} \quad k^e = \int_{\Omega^e}\underline{B}^{e^T}\,\underline{C}\,\underline{B}^e\,dV$$

(68) $\underline{r}^e = \int_{\partial\Omega^e}\underline{\Phi}^{e^T}\,\underline{t}\,ds$ $\qquad \underline{q}^e = -\int_{\Omega^e}\underline{\Phi}^{e^T}\underline{b} - \int_{\partial\Omega^e}\underline{\Phi}^{e^T}\hat{\underline{t}}$

Essendo β arbitrario, $\boxed{\underline{k}^e \underline{s}^e + \underline{q}^e = \underline{\pi}^e}$

Possiamo raggruppare i termini per nodi:

$$\begin{bmatrix} \underline{k}_{ii}^e & \underline{k}_{is}^e & \underline{k}_{in}^e & \underline{k}_{ie}^e \\ \underline{k}_{si}^e & \underline{k}_{ss}^e & \underline{k}_{sn}^e & \underline{k}_{se}^e \\ \underline{k}_{ni}^e & \underline{k}_{ns}^e & \underline{k}_{nn}^e & \underline{k}_{ne}^e \\ \underline{k}_{ei}^e & \underline{k}_{es}^e & \underline{k}_{en}^e & \underline{k}_{ee}^e \end{bmatrix} \begin{bmatrix} \underline{s}_{(i)} \\ \underline{s}_{(s)} \\ \underline{s}_{(n)} \\ \underline{s}_{(e)} \end{bmatrix} + \begin{bmatrix} \underline{q}_{(i)}^e \\ \underline{q}_{(s)}^e \\ \underline{q}_{(n)}^e \\ \underline{q}_{(e)}^e \end{bmatrix} = \begin{bmatrix} \underline{\pi}_{(i)}^e \\ \underline{\pi}_{(s)}^e \\ \underline{\pi}_{(n)}^e \\ \underline{\pi}_{(e)}^e \end{bmatrix}$$ quindi

$$\underline{\pi}_{(i)}^e = \underline{k}_{ii}^e \underline{s}_{(i)} + \underline{k}_{is}^e \underline{s}_{(s)} + \underline{k}_{in}^e \underline{s}_{(n)} + \underline{k}_{ie}^e \underline{s}_{(e)} + \underline{q}_{(i)}^e$$

Per l'equilibrio si ha $\underline{F}_{(i)} = \sum_e \underline{\pi}_{(i)}^e$

Si costruisce quindi matrice $\boxed{\underline{k}\,\underline{s} + \underline{q} = \underline{F}}$ dove l'incognita è $\underline{s}$.

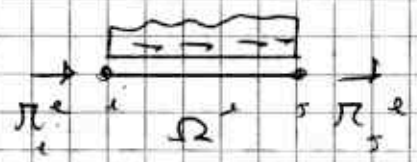
Supponiamo i collegam. solo nei nodi, approx

Possiamo arricchire scegliendo + nodi.

Carichi esterni assegnati di solito si riducono ad azioni nodali.

Ex: trave con carico assiale

Cons. elemento Ω^e:



$u(0)=0$ $\quad$ $EsW'(\ell)=F$

$$\boxed{W^e(z) = W_i\,\phi_i^e(z) + W_j\,\phi_j^e(z)}$$

Sappiamo che $\int_0^\ell EsW'\eta' - \int_0^\ell p\,\eta - F\,\eta(\ell) = 0$

Poniamo $\phi_i = a_i z + b_i$ e $\phi_j = a_j z + b_j$

Deve essere $\phi_i(z_i)=1$, $\phi_j(z_i)=0$ e $\phi_i(z_j)=0$, $\phi_j(z_j)=1$

Allora:

$$\phi_i(z) = \frac{z - z_j}{z_i - z_j} \; ; \quad \phi_j(z) = \frac{z - z_i}{z_j - z_i}$$

Introduciamo $\varepsilon^e = D u^e = \frac{\partial}{\partial z}\left(w_i \phi_i + w_j \phi_j\right) = w_i \phi_i' + w_j \phi_j' = \frac{w_j - w_i}{z_j - z_i}$

Introduciamo $\sigma^e = E \varepsilon^e$.

Poniamo $\eta^e(z) = \beta_i \phi_i^e(z) + \beta_j \phi_j^e(z)$

Equil.: $\int_{z_i}^{z_j} E A u' \eta' - \int_{z_i}^{z_j} p \eta - \Pi_i^e \eta(z_i) - \Pi_j^e \eta(z_j) = 0$

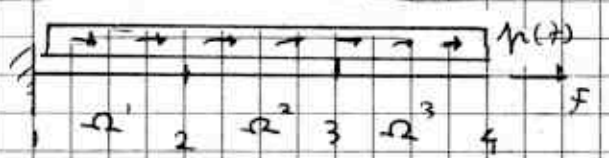
Sostituendo: $k_{ii}^e w_i \beta_i + k_{ij}^e w_i \beta_j + k_{ji}^e w_j \beta_i + k_{jj}^e w_j \beta_j + q_i^e \beta_i + q_j^e \beta_j = \Pi_i^e \beta_i + \Pi_j^e \beta_j$

con $k_{ab}^e = \int_{z_i}^{z_j} E A \phi_a' \phi_b' \, dz$, $q_a^e = -\int_{z_i}^{z_j} p \phi_a \, dz$.

Scelta: $(\beta_i = 1, \beta_j = 0)$ e $(\beta_i = 0, \beta_j = 1)$

$$\begin{cases} k_{ii}^e w_i + k_{ij}^e w_j + q_i^e = \Pi_i^e \\ k_{ji}^e w_j + k_{jj}^e w_j + q_j^e = \Pi_j^e \end{cases}$$

Definiamo le ϕ x tutta la trave e usiamo Ritz o Galerkin: conv. x sempl. p unif. Equil.:



$$\begin{cases} k_{11}^{(1)} w_1 + k_{12}^{(1)} w_2 + q_1^{(1)} = \Pi_1^{(1)} \\ k_{21}^{(1)} w_1 + k_{22}^{(1)} w_2 + q_2^{(1)} = \Pi_2^{(1)} \end{cases}$$

$$\begin{cases} k_{22}^{(2)} w_2 + k_{23}^{(2)} w_3 + q_2^{(2)} = \Pi_2^{(2)} \\ k_{32}^{(2)} w_2 + k_{33}^{(2)} w_3 + q_3^{(2)} = \Pi_3^{(2)} \end{cases}$$

$$\begin{cases} k_{33}^{(3)} w_3 + k_{34}^{(3)} w_4 + q_3^{(3)} = \Pi_3^{(3)} \\ k_{43}^{(3)} w_3 + k_{44}^{(3)} w_4 + q_4^{(3)} = \Pi_4^{(3)} \end{cases}$$

$\Pi_2^{(1)} + \Pi_2^{(2)} = 0$

$\Pi_3^{(2)} + \Pi_3^{(3)} = 0$

$\Pi_4^{(3)} = F$

C. essenz.:

(7a) $w_1 = 0 \Rightarrow$ cancelliamo la 1.

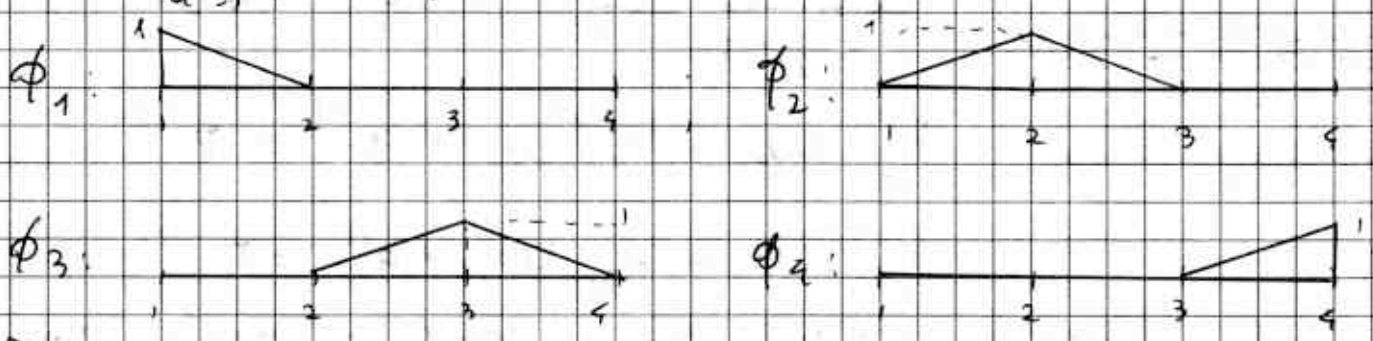
$$\left(k_{22}^{(1)} + k_{22}^{(2)}\right) w_2 + k_{23}^{(2)} w_3 = -\left(q_2^{(1)} + q_2^{(2)}\right)$$

$$k_{32}^{(2)} w_2 + \left(k_{33}^{(2)} + k_{33}^{(3)}\right) w_3 + k_{34}^{(3)} w_4 = -\left(q_3^{(2)} + q_3^{(3)}\right)$$

$$k_{43}^{(3)} w_3 + k_{44}^{(3)} w_4 = F - q_4^{(3)}$$

Possiamo definire le ϕ_k con $k = 1 \dots 4$ su tutta la trave

$$w = \sum_{a=1}^{4} w_a \phi_a(z)$$



Prendiamo la C. essenz.:

$w = \sum_{a=2}^{4} w_a \phi_a$. Usiamo Ritz:

$$\Pi = \frac{1}{2}\int_0^l EJ (w'')^2 - \int_0^l p\, w - F\, w(l)$$

$$\tilde{\Pi} = \frac{1}{2}\sum_{a=2}^{4}\sum_{b=2}^{4} w_a w_b \int_0^l EJ\, \phi_a'' \phi_b'' - \sum_{b=2}^{4} w_b \int_0^l p\, \phi_b - F w_4$$

Deriv. risp. a param.:

$$\frac{\partial \tilde{\Pi}}{\partial w_B} = \sum_{a=2}^{4} w_a \int_0^l EA\, \phi_a'' \phi_b'' - \int_0^l p\, \phi_b - F \delta_{4b}$$

$$\boxed{\sum_{a=2}^{4} k_{ba} w_a + q_B = F \delta_{4b}}$$, $b = 2,3,4$ con

$$k_{ba} = \int_0^l EJ\, \phi_b'' \phi_a'' \quad ; \quad k_{22} = k_{22}^{(1)} + k_{22}^{(2)}, \quad k_{21} = k_{21}^{(1)}$$

Ora usiamo Galerkin.

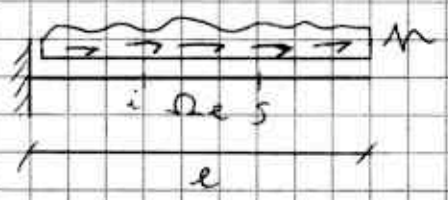
$$\eta = \sum_{b=2}^{4} w_B \phi_B(t) \quad ; \quad \int_0^l EJ w'\eta' - \int_0^l p\eta - F\eta(l) = 0$$

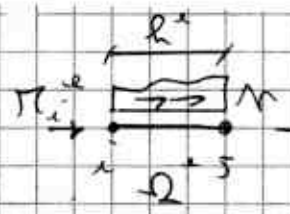
$$\sum_{a=2}^{4}\sum_{b=2}^{4} w_a \beta_b \int_0^l EJ \phi_a' \phi_b' - \sum_{b=2}^{4} \beta_b \int_0^l p\phi_b - F\beta_4 = 0$$

Prendiamo $\beta_2 = 1$, $\beta_3 = \beta_4 = 0$, abbiamo stesse eq. di prima.

6/04/09

Sia F.E.M.





Eq. equil.:

$$K_{ii}^{(e)} w_i + K_{ij}^{(e)} w_j + q_i^{(e)} = R_i^{(e)}$$

$$K_{ji}^{(e)} w_i + K_{jj}^{(e)} w_j + q_j^{(e)} = R_j^{(e)}$$

Si combina tramite equil tra i nodi + equil. tratte

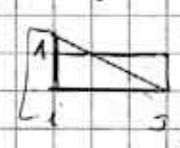
$w^{(e)} = w_i \phi_i(z) + w_j \phi_j(z)$. Possono assumere lin in z [$\int_{\Omega^e} EJ w'\eta' - \int_{\Omega^e} p\eta - R_i^{(e)} w_i - R_j^{(e)} w_j = 0$, richiesta max. deriv. prima]. Per avere senso in i devo avere $\phi_i = \frac{z - z_j}{z_i - z_j}$ e $\phi_j = \frac{z - z_i}{z_j - z_i}$. Sost.:

$$K_{ij} = \int_0^{h^e} EA\, \phi_i^{(e)\prime} \phi_j^{(e)\prime}\, dz = \pm \frac{EA}{h^e}$$

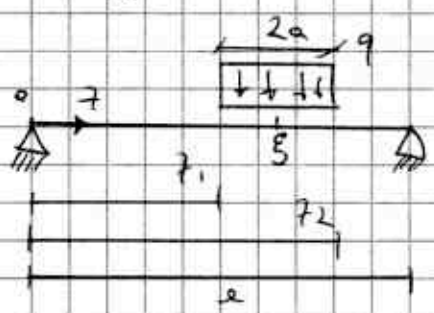
(sempre EA cost lungo la tratte)

$$\phi_i^{(e)\prime} = \frac{-1}{h^e}, \quad \phi_j^{(e)\prime} = \frac{1}{h^e}$$

$$q_i^{(e)} = -\int_0^{h^e} p\, \phi_i^{(e)}, \quad \text{se } p \text{ cost } h_0 = -p \frac{1}{2} h^e$$



Ex:



$$\tilde{\Pi} = \frac{1}{2}\int_0^l EI (v'')^2 - \int_{z_1}^{z_2} q v$$

Vogliamo sol. approx con Ritz.

Cond. essenziali: (nello spost., nel Π è c. naturale)

$v(0)=0$, $v(l)=0$ sono omogenee, quindi $v(z)=\sum_{k=1}^{N}\alpha_k \sin\frac{k\pi z}{l}$

$v''(z)=-\sum_{k=1}^{N}\alpha_k k^2\frac{\pi^2}{l^2}\sin\frac{k\pi z}{l}$. Sost. in Π:

$$\Pi=\frac{1}{2}\sum_{h=1}^{N}\sum_{k=1}^{N}\alpha_h\alpha_k\frac{\pi^4}{l^4}k^2h^2\int_0^l\sin\left(\frac{h\pi z}{l}\right)\sin\left(\frac{k\pi z}{l}\right)-\sum_{h=1}^{N}\alpha_h\cdot$$

$$\cdot\int_{\xi-a}^{\xi+a}q\sin\left(\frac{h\pi z}{l}\right)$$

[RICORDA: PUÒ SEMPRE I DIFFERENZIARE NELL' INTEGR.] [porto $\xi=\frac{z_1+z_2}{2}$; $a=\frac{z_2-z_1}{2}$]

[$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$; $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$;

$\sin\alpha\sin\beta=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$]

$$\int_0^l\sin\frac{h\pi z}{l}\sin\frac{k\pi z}{l}=\frac{1}{2}\int_0^l\left(\cos\frac{(h-k)\pi z}{l}-\cos\frac{(h+k)\pi z}{l}\right)=$$

- $h\neq k$:

$$=-\frac{1}{2}\left[\frac{l}{(h-k)\pi}\sin\frac{(h-k)\pi z}{l}-\frac{l}{(h+k)\pi}\sin\frac{(h+k)\pi z}{l}\right]_0^l=0$$

- $h=k$:

$$=\frac{1}{2}\int_0^l\left(1-\cos\frac{2\pi hz}{l}\right)=\frac{1}{2}\left[z-\frac{l}{2h\pi}\sin\frac{2h\pi z}{l}\right]_0^l=\frac{l}{2}$$

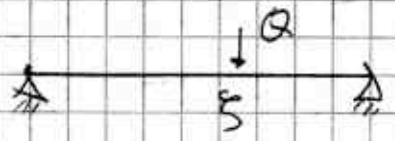
Quindi $\Pi=\frac{1}{2}\sum_{h=1}^{N}D_{hh}\alpha_h\alpha_h+\sum_{h=1}^{N}q_h\alpha_h$

porto $D_{hh}=\frac{\pi^4h^4}{l^3}\frac{l}{2}$; a memoria: $\int_{\xi-a}^{\xi+a}\sin\frac{h\pi z}{l}=\left[-\frac{l}{h\pi}\cos\frac{h\pi z}{l}\right]_{\xi-a}^{\xi+a}=-\frac{l}{h\pi}\left(\cos\frac{(\xi+a)h\pi}{l}-\cos\frac{(\xi-a)h\pi}{l}\right)=$

$=2\frac{l}{h\pi}\sin\frac{\xi h\pi}{l}\cdot\sin\frac{ah\pi}{l}$; poniamo $q_h=\int_{\xi-a}^{\xi+a}q\sin\frac{h\pi z}{l}$.

$$\frac{\partial\Pi}{\partial\alpha_k}=D_{kk}\alpha_k+q_k=0\rightsquigarrow\alpha_k=-\frac{q_k}{D_{kk}}$$

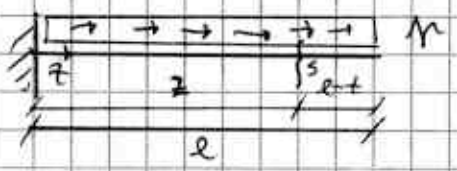
Ex:



$$\tilde{\Pi} = \frac{1}{2}\int EI\, v'' - Q\, v(\xi)$$ [basta valutare, vari sin $\frac{2\pi\xi}{l}$ per ξ e sost. in q_n]

CONFRONTO FEM con SOLUZIONE ESATTA

Ex:



$N(z) = p(l-z)$; $EA\, w' = N$

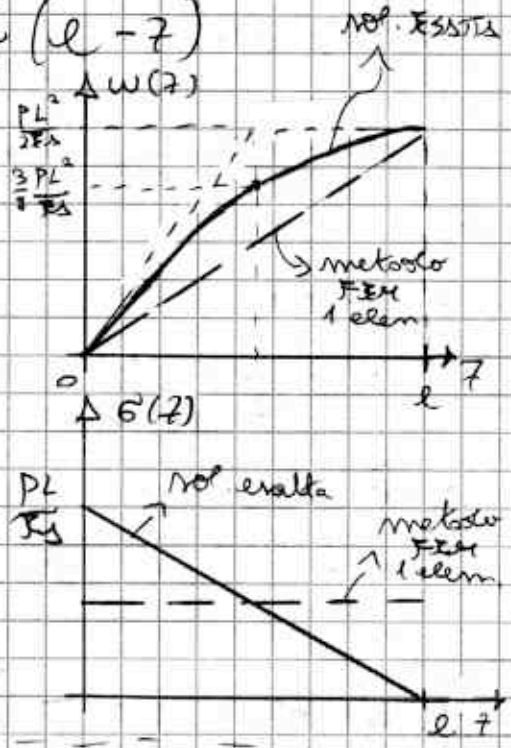
$w' = \frac{N}{EA} = \frac{p}{EA}(l-z)$

$w = \frac{p}{EA}\left(lz - \frac{z^2}{2}\right) + C \rightarrow w(0) = 0 \Rightarrow C = 0$

espr. esatta spost. su asse trave.

$\sigma(z) = \frac{N(z)}{A} = \frac{p}{A}(l-z)$

$w(l) = \frac{pl^2}{2EA}$; $w\left(\frac{l}{2}\right) = \frac{3}{8}\frac{pl^2}{EA}$



Ora con FEM, modello a 1 elemento

$k_{11}^{(1)} w_1 + k_{12}^{(1)} w_2 + q_1^{(1)} = r_1^{(1)}$

$k_{21}^{(1)} w_1 + k_{22}^{(1)} w_2 + q_2^{(1)} = r_2^{(1)}$

$k_{11}^{(1)} = k_{22}^{(1)} = -k_{12}^{(1)} = -k_{21}^{(1)} = \frac{EA}{l}$

$q_1^{(1)} = q_2^{(2)} = -\frac{pl}{2}$. Impongo c. essenz.: $w_1 = 0$

Cancello eq. nodo 1 e la sost. con

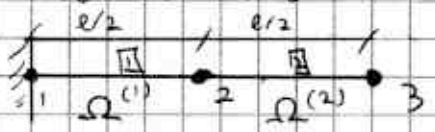
Essendo $r_2^{(1)} = 0$ ho $k_{21}^{(1)} w_1 + k_{22}^{(1)} w_2 + q_2^{(1)} = 0$

Quindi $w_2 = -\frac{q_2^{(1)}}{k_{22}^{(1)}} = \frac{pl^2}{2EA} \equiv w(l)$ in sol. esatta!

$w^{(e)} = \cancel{w_1\left(1 - \frac{z}{l}\right)} + w_2 \frac{z}{l}$; spost. è LINEARE!

$$\sigma_z^{(1)} = E\,\varepsilon_z^{(1)} = E\,\frac{u_2-u_1}{l} = \frac{pl}{2EA}$$

Ora conv. FEM a 2 elem.



$$\text{I}\begin{cases} k_{11}^{(1)} u_1 + k_{12}^{(1)} u_2 + q_1^{(1)} = R_1^{(1)} & [\text{eq. nodo } 1] \\ k_{21}^{(1)} u_1 + k_{22}^{(1)} u_2 + q_2^{(1)} = R_2^{(1)} & [\text{eq. nodo } 2] \end{cases}$$

$$\text{II}\begin{cases} k_{22}^{(2)} u_2 + k_{23}^{(2)} u_3 + q_2^{(2)} = R_2^{(2)} & [\text{eq. nodo } 2] \\ k_{32}^{(2)} u_2 + k_{33}^{(2)} u_3 + q_3^{(2)} = R_3^{(2)} & [\text{eq. nodo } 3] \end{cases}$$

$$\text{Globale}\begin{cases} \cancel{k_{11}^{(1)} u_1 + k_{12}^{(1)} u_2 + q_1^{(1)} = R_1^{(1)}} \\ k_{21}^{(1)} u_1 + \left(k_{22}^{(1)} + k_{22}^{(2)}\right) u_2 + k_{23}^{(2)} u_3 + q_{(2)}^{1} + q_{(2)}^{2} = \underbrace{R}_{R_2^{(1)}+R_2^{(2)}} \\ k_{32}^{(2)} u_2 + k_{33}^{(3)} u_3 + q_3^{(2)} = \underbrace{0}_{R_3^{(2)}} \end{cases}$$

$$k_{11}^{(1)} = \underset{k_{22}^{(2)}}{k_{22}^{(1)}} = k_{33}^{(2)} = \frac{EA}{l/2} = -k_{12}^{(1)} = -k_{21}^{(1)} = -k_{23}^{(2)} = -k_{32}^{(2)}$$

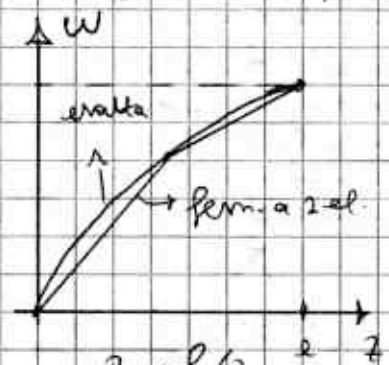
$$q_1^{(1)} = q_2^{(1)} = q_2^{(2)} = q_3^{(2)} = -\frac{pl}{4}.$$

Cond.: $u_1 = 0$

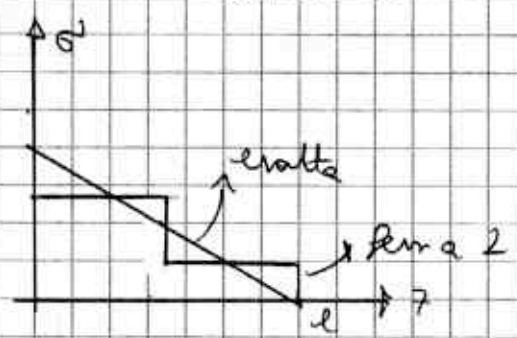
Sol: $u_1 = 0$

$u_2 = \frac{3}{8}\frac{pl^2}{EA}$

$u_3 = \frac{pl^2}{2EA}$



Spost. giusti, ma andam. lineare.



$$u^{(2)} = u_2\,\frac{z-l}{l/2-l} + u_3\,\frac{z-l/2}{l-l/2}$$

$$\sigma_z^{(1)} = E\,\frac{u_2-u_1}{l/2} = \frac{3}{4}\frac{PL}{EA}$$

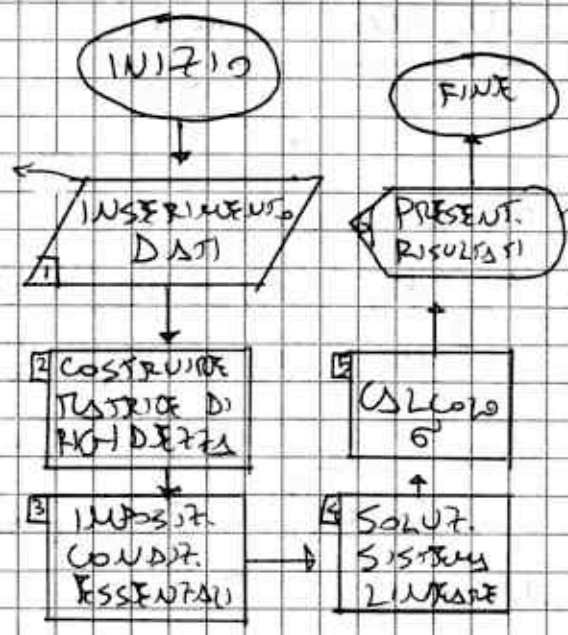
$$\sigma_z^{(2)} = E\,\frac{u_3-u_2}{l/2} = \frac{1}{4}\frac{PL}{A}$$

$\Downarrow$

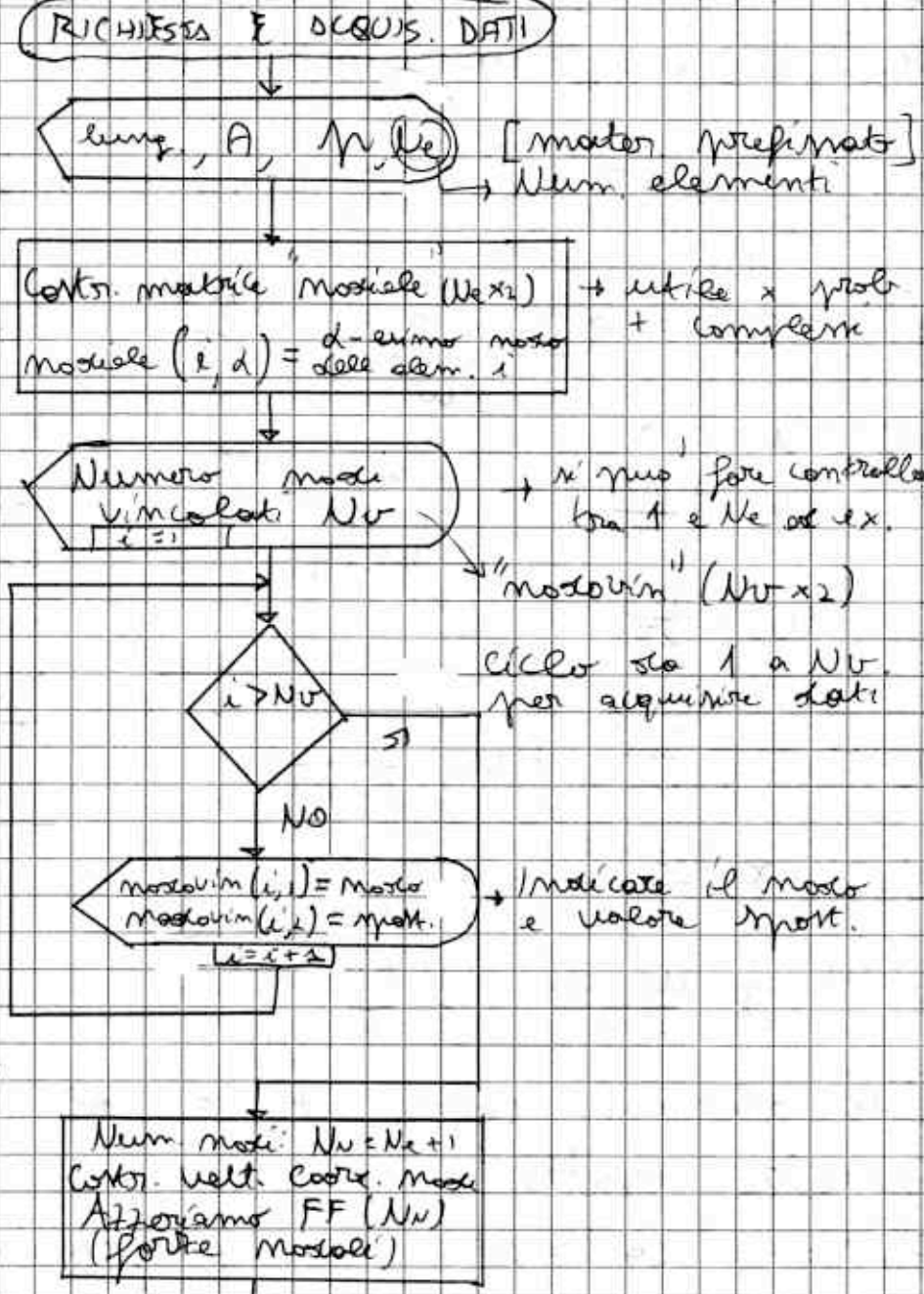
Più elementi conv., + la sol. approx. tende a quella esatta.

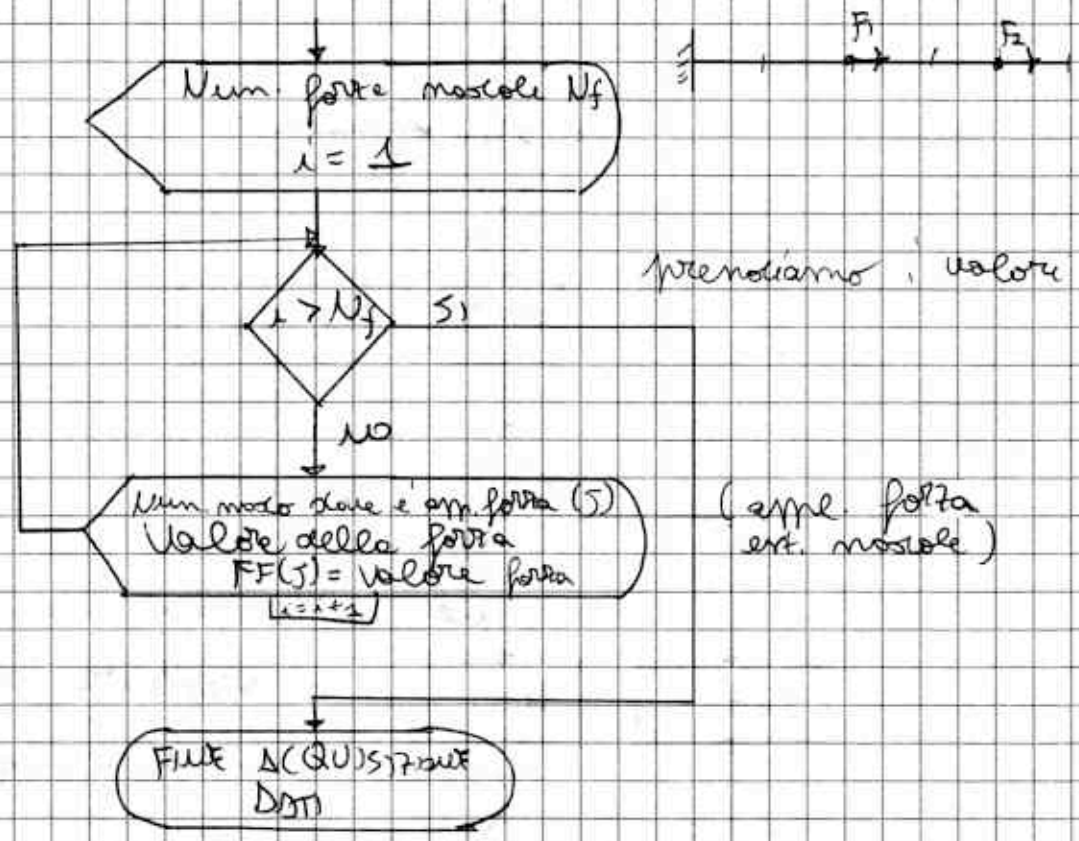
Possiamo costruire un programma per automatizzare il processo.

IMPLEMENTAZIONE PROGRAMMA PER LO STUDIO DGL FLETTENTI FINITI:



1 : Dettaglio:



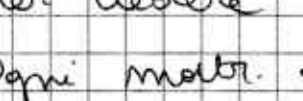


07/04/09

2 COSTRUZIONE MATRICE DI RIGIDEZZA

Partiamo dal nodo e passiamo in rassegna gli elementi per vedere quale contiene quel nodo.

Ogni matr. di rigidezza ha 2 eq.: si prende la prima se il nodo è il primo dell'elemento; si prende la 2 se il nodo è il 2 dell'elemento



$$\left.\begin{aligned} &1)\ k_{hh}^{e} s_h + k_{hi}^{e} s_i + q_h^{e} = r_h^{e} \\ &2)\ k_{ih}^{e} s_h + k_{ii}^{e} s_i + q_i^{e} = r_i^{e} \end{aligned}\right\} \text{eq. elemento } \Omega^{\circ}$$

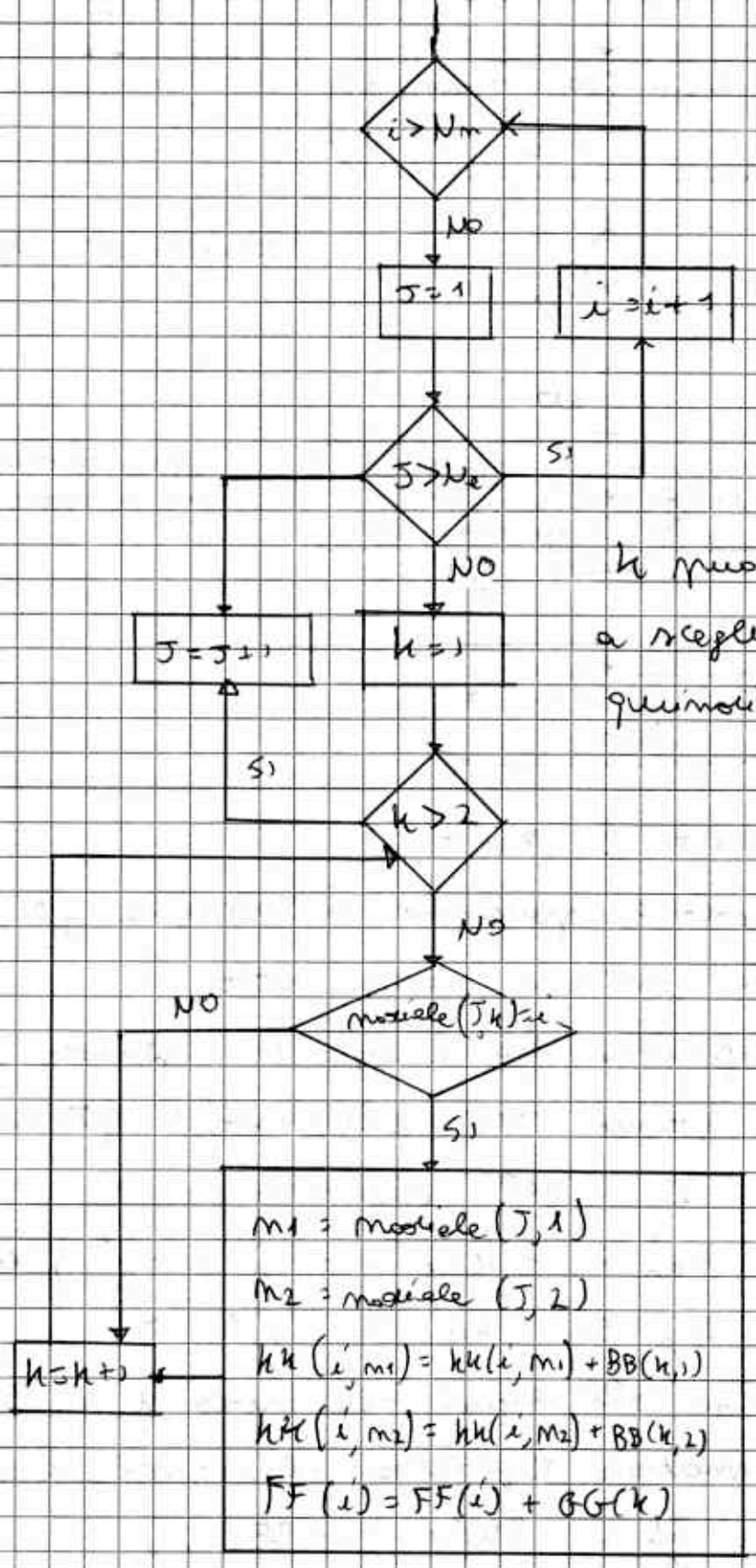
L'eq. che contribuisce all'equil. del nodo i è la 2)

Costruiamo BB = matrice rigidezza elemento; l_0 = lunghezza elemento

$BB(1,1) = BB(2,2) = \frac{EA}{l_0} = -BB(1,2) = -BB(2,1)$

$BB(1) = GG(2) = \frac{\gamma A l_0}{2}$ (matrice dei carichi)

BB(2x2): matrice rig. elem.
GG(2): carichi elem.
KK ($N_m \times N_m$)
KK(i,j) = 0; i = 1



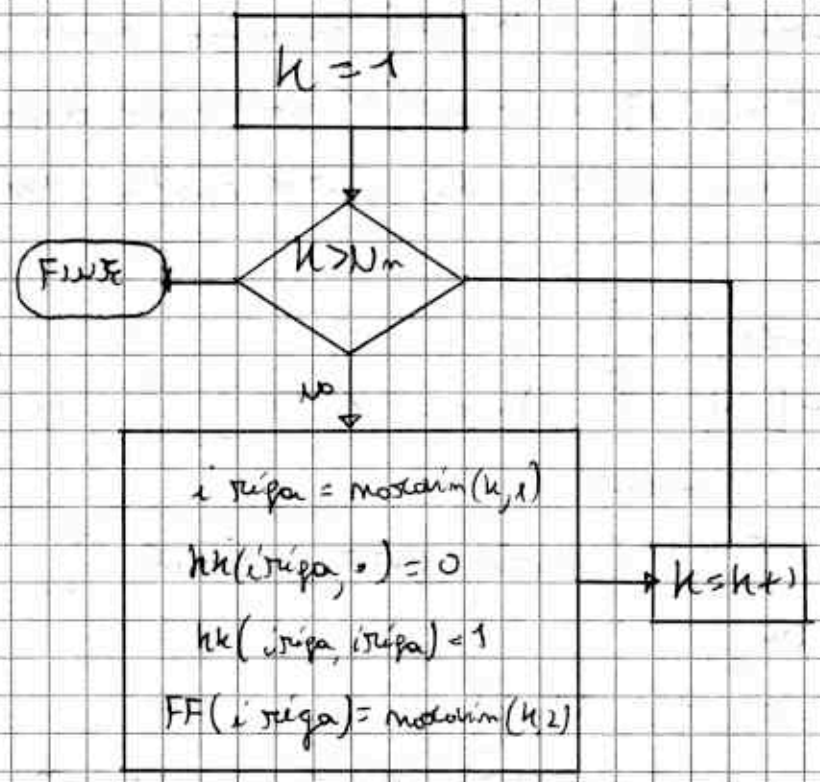
k può assumere valori utile a scegliere il modo e quindi l'equazione

3 IMPOSIZIONE CONDIZIONI ESSENZIALI

Abbiamo sistema di N equaz. in N incognite

Facciamo ciclo per esplorare questa matrice

(→)



$$\sigma_z^e = E \frac{w_j - w_i}{l_e}$$ → Calcolo tensioni

•

ISTRUZIONI DEL PROGRAMMA MATHEMATICA

```
lin = Input["lunghezza della trave (cm)"];
A = Input["Area della sezione (cm²)"];
Em = 210000; (* modulo elastico acciaio *)
p = Input["carico assiale uniforme (kg/cm)"];
Ne = Input["numero degli elementi"];
Array[nodiele(Ne, 2)];
Do[{nodiele[i,1] = i, nodiele[i,2] = i+1}, {i, Ne}]
```

Dobbiamo acquisire i nodi vincolati

```
Nv = Input["numero dei nodi vincolati"];
```

Def. matrice in cui metterli

```
Array[nodovin, {Nv, 2}];
```

```
Do[{nodovin[i,1] = Input["numero nodo vincolato"],
    nodovin[i,2] = Input["Spostamento assegnato (cm)"]}, {i,Nv}]
```

Mettiamo in un vettore le coordinate dei nodi

```
Nm = Ne + 1
Array[znodo, Nm];
Do[{znodo[i] = (i-1) × lim / Ne}, {i, Nm}]
```

Vogliamo f. nodali

```
Nf = Input["Numero forze nodali"];
Array[FF, Nm],
Do[FF[i] = 0, {i, Nm}]
```

Accumuliamo nel vettore le f. nodali che abbiamo assegnato.

```
Do[{i for = Input["numero nodo su cui agisce la forza"],
    FF(i for) = Input["intensità della forza (kg)"]}, {i,Nf}]
```

Definiamo il vettore per iniziare la COSTRUZIONE della MATRICE DI RIGIDEZZA.

```
Array[BB, {2,2}];
BB[1,1] = Ne × Em × A / lim
BB[2,2] = Ne × Em × A / lim
BB[1,2] = -Ne × Em × A / lim
BB[2,1] = -Ne × Em × A / lim
Array[GG, 2];
GG[1] = - m × lim / (2 × Ne)
GG[2] = - m × lim / (2 × Ne)
Array[kk, {Nm, Nm}],
Do[Do[kk(i,j) = 0, {j,Nm}], {i, Nm}]
Do[Do[Do[If[modiele[j,k] = i, {n1 = modiele[j,1],
```

(i nodo
j elemento
k-esimo nodo di j)

nodo elem. k=1 k=2

MATRICE

```
m2 = modulo[j, 2], kk[i, m1] = kk[i, m1] + BB[k, 1],
kk[i, m2] = kk[i, m2] + BB[k, 2], FF[i] = FF[i] - GG[k]}],
{k, 2}, {j, Ne}], {i, Nm}]
```

DI RIGHETTA

Imponiamo le condizioni essenziali

```
Do[{i riga = mostovin[k, 1], Do[kk[i riga, j] = 0 {j, Nm}],
    kk[i riga, i riga] = 1, FF[i riga] = mostovin[k, 2]}, {k, Nv}]
```

Vogliamo soluzione

```
u = Inverse[Array[kk, {Nm, Nm}]] . Array[FF, {Nm}]
Array[tensione, Ne];
Do[tensione[i] = Em × (u[[i + 1]] - u[i]) × Ne/lm, {i, Ne}]
```

FINE PARTE 1 (07/04/09)

TEORIA DELLE PIASTRE - Prof. Mario Lemmo - 6 CFU - A.A. 2008/09

Parte 2 (da 20/04/09 a 26/5/09)

82. INTRODUZIONE ALLE PIASTRE

84. SFORZI RISULTANTI / FORZE MEMBRANALI - TAGLIANTI

85. MOMENTI FLETTENTI - TORCENTI

86. PROBLEMA BIDIMENSIONALE DEFORMAZIONE SU SUPERFICIE MEDIA

89. EQUAZIONI COSTITUTIVE

96. CAMBIAMENTO DI COORDINATE

98. TEORIA DI KIRCHHOFF - LOVE (o TEORIA CLASSICA DELLE PIASTRE)

99. VINCOLI INTERNI / SFORZI ATTIVI - REATTIVI

103. RIGIDEZZA FLESSIONALE

104. EQUAZIONI DI EQUILIBRIO PIASTRA

111. FORMULAZIONE VARIAZIONALE DEL PROBLEMA DI EQUILIBRIO

114. SOLUZIONI PER L'EQUILIBRIO DI PIASTRE RETTANGOLARI NELLA TEORIA DI KIRCHHOFF

124. PIASTRE CIRCOLARI DI KIRCHHOFF - LOVE

124. PASSAGGIO IN COORDINATE CILINDRICHE

128. SIMMETRIA ASSIALE

131. ESEMPI DI PIASTRE CIRCOLARI

136. STUDIO DEL PROBLEMA MEMBRANALE CON GLI ELEMENTI FINITI

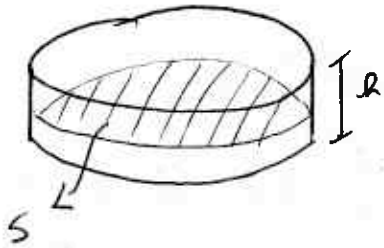
144. TEORIA DI REISSNER - MINDLIN

152. PIASTRE CIRCOLARI NELLA TEORIA DI REISSNER - MINDLIN

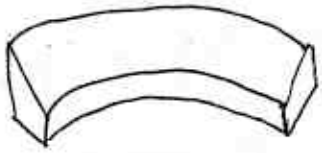
158. (da STUDIO DEL PROBLEMA MEMBRANALE CON GLI ELEMENTI FINITI)

INTRODUZIONE ALLE PIASTRE

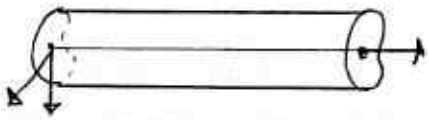
20/4/2009

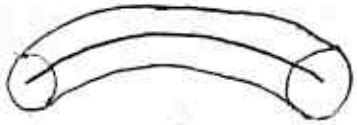


Piastra è cilindro con h piccolo risp a dimens. base.



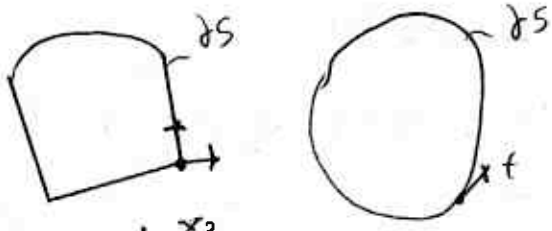
Δ trave corrisp. trave curvilinea.



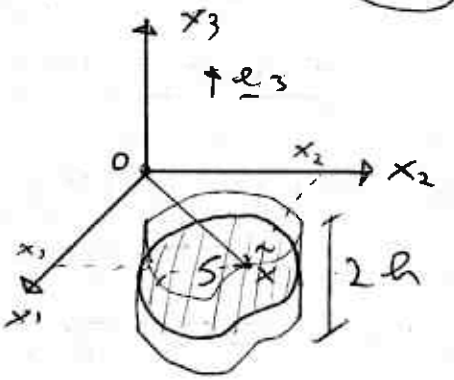


Δ piastra con il guscio

S: "piano medio" può essere delim. da contorno regolare o regolare a tratti. Regolare: in ogni punto è definito il vett. tang. $\underline{t}$.



Va bene anche ∂S regolare a tratti.



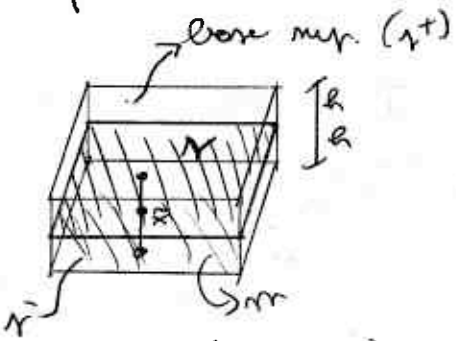
$\tilde{x} \in$ piano S

$\tilde{x} = O + x_1 \underline{e}_1 + x_2 \underline{e}_2 = O + x_\alpha \underline{e}_\alpha$

($\alpha = 1,2$; $i, j, k = 1,2,3$. convenz.)

Il cilindro (piastra) è def. come:

$$P = \{ x : x = \tilde{x} + x_3 \underline{e}_3 \ \tilde{x} \in S, \ x_3 \in (-h, h) \}$$

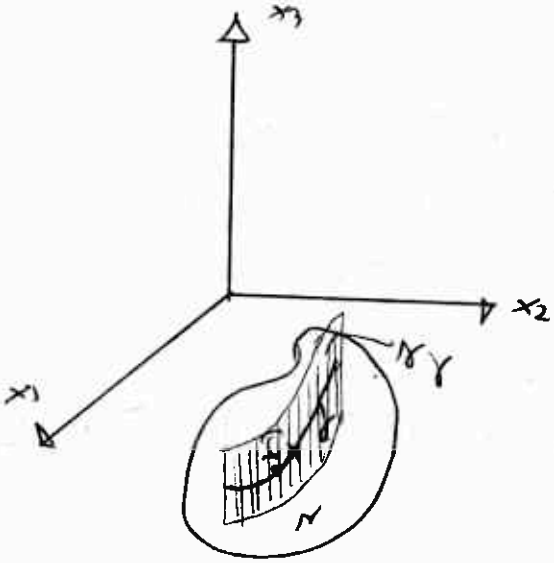


Cilindro è inv. di segmenti detti FIBRE (o fibre trasversali) [si identifica corpo con config. geometrica che occupa certa regione in certa configurazione; corpo non è solo geometrico, è elem. materiale, ma comodo x semplicità ident. piastra con regione occupata, cioè cilindro

in config. indeformata.]

- Base superiore (s^+) : $s^+ = \{ x \in P : x_3 = h \}$
- " inferiore (s^-) : $s^- = \{ x \in P : x_3 = -h \}$
- Sup. laterale detta MANTELLO : $m = \{ x \in P : \tilde{x} \in \partial S$

Si vuole ridurre prob. equilibrio da prob. 3D a 2D (sup. media) [trace: da 3D a 1D].



Cons. curva γ su sup. media s parametr. da lungh. arco:

$$\gamma : \begin{cases} x_1 = x_1(s) \\ x_2 = x_2(s) \end{cases}$$

$$x = O + x_1 \underline{e}_1 + x_2 \underline{e}_2 \in \gamma$$

Deriv. risp. a param. ho vett. tan.:

$$\underline{\tau} = \frac{dx_1}{ds} \underline{e}_1 + \frac{dx_2}{ds} \underline{e}_2 \quad |\underline{\tau}| = 1$$

(vero solo se il param è param "lungh. d'arco")

Cons. superficie è piatta che cons. tutte le fibre passanti per γ, sez. piastra, detta s_γ

$$s_\gamma : \begin{cases} x_\alpha = x_\alpha(s) \\ x_3 = t \end{cases} \quad \text{Se } s \in I, \quad (s, t) \in I \times (-h, h)$$

[caso particolare, perché in generale è eq. di una sup. in $\mathbb{R}^3$ è $x_i = x_i(s, t)$]

Vogliamo vett tan. alle curve a param. costante.

$\underline{x} = x_i(s, t)\, \underline{e}_i$ in generale.

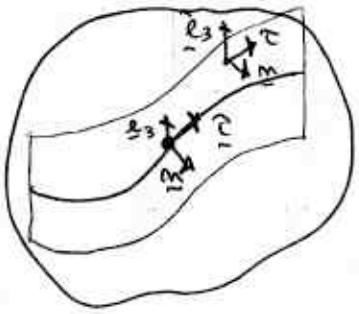
$\underline{x}_{,s} = x_{i,s}(s, t)\, \underline{e}_i$

83 $\underline{x}_{,t} = x_{i,t}(s, t)\, \underline{e}_i$

Qui: $\underline{x} = x_\alpha(r)\,\underline{e}_\alpha + t\,\underline{e}_3$

$\underline{x}_{,r} = x_{\alpha,r}\,\underline{e}_\alpha = \underline{\tau}$ [curve // a γ, incluso]

$\underline{x}_{,t} = \underline{e}_3$ [la fibra]



La normale è data da $\underline{n} = \underline{\tau} \times \underline{e}_3$

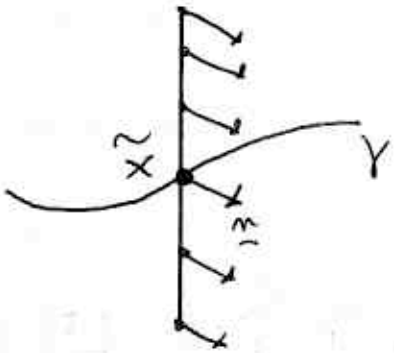
Se ci muoviamo lungo una fibra!

$\underline{\tau} = \frac{dx_\alpha}{dr}(r)\,\underline{e}_\alpha$

$\underline{n} = \underline{\tau}(r) \times \underline{e}_3 = \underline{n}(r) \Rightarrow$ NON USARLA

Tutti i punti di una fibra hanno stessa normale

Fibra:



In tutti i punti è ocef.

$\underline{T}\,\underline{n} = \underline{t}_{\underline{n}}$.

Com. il risultante delle tensioni su una fibra $(\underline{f})$

$$\underline{f} = \int_{-h}^{h} \underline{t}_{\underline{n}}\,dx_3 = \int_{-h}^{h} \underline{T}\,\underline{n}\,dx_3 =$$

$$= \left(\int_{-h}^{h} \underline{T}\,dx_3\right)\underline{n} = \underline{N}\,\underline{n}$$

, tolta la dipendenza da x_3.

(integr. tensore e tensore)

SFORZO RISULTANTE

- $N_{\alpha\beta}$: FORZE MEMBRANALI: $N_{11}, N_{12}, N_{21}, N_{22}$
- $N_{3\alpha}$: FORZE TAGLIANTI (Q_α): $N_{31} = Q_1$, $N_{32} = Q_2$

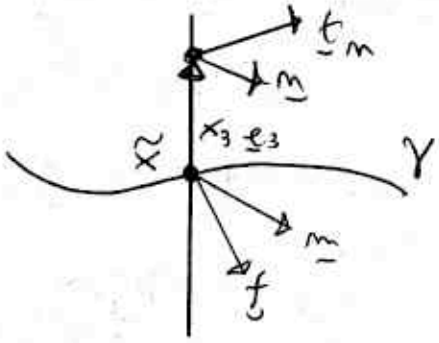
$\underline{N}\,\underline{n} = \underline{N}\,(n_\beta\,\underline{e}_\beta)$; $\underline{n} = \underline{\tau} \times \underline{e}_3$, quindi

$n_3 = \underline{n} \cdot \underline{e}_3 = 0$ e allora N_{33} non c'è.

$= N_{i\beta}\,\underline{e}_i\,n_\beta$

Il momento risultante:

$$\underline{m} = \int_{-h}^{h} x_3 \, \underline{e}_3 \times \underline{t}_n \, dx_3 =$$



$$= \int_{-h}^{h} x_3 \, \underline{e}_3 \times \underline{T} \, \underline{n} \, dx_3 =$$

$$= \underline{e}_3 \times \left(\int_{-h}^{h} \underline{T} \, x_3 \, dx_3 \right) \underline{n} = \underline{e}_3 \times \overset{\underline{c}}{\underline{M} \, \underline{n}} =$$

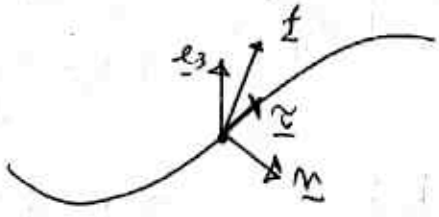
$$= M_{i\beta} N_\beta \, \underline{e}_i = M_{\alpha\beta} N_\beta \, e_\alpha$$ (la comp. 3 non interv.)

- M_{11} e M_{22}: MOMENTI FLETTENTI
- $M_{12} = M_{21}$: MOMENTI TORCENTI

$\underline{N}$, $\underline{M}$: SFORZI RISULTANTI

$\underline{f}$ è una forza per un. di lung; $\underline{m}$ è "forza" $[F \cdot L \cdot \frac{1}{L}]$
mom. risult per unità di lungh. di γ.

$$\underline{f} = \underline{N} \, \underline{n} \; ; \quad \underline{m} = \underline{e}_3 \times \underline{M} \, \underline{n}$$

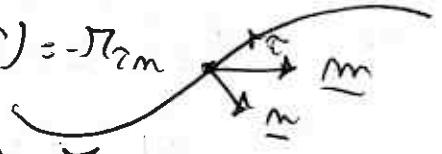


$$\left.\begin{aligned} \underline{f} \cdot \underline{n} &= \underline{N} \, \underline{n} \cdot \underline{n} = N_{nn} \\ \underline{f} \cdot \underline{\tau} &= \underline{N} \, \underline{n} \cdot \underline{\tau} = N_{\tau n} \end{aligned}\right\} \text{membranali}$$

[ricorda: $A_{ij} = \underline{A} \, \underline{e}_j \cdot \underline{e}_i$]

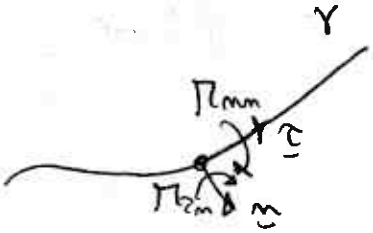
$$\left. \underline{f} \cdot \underline{e}_3 = \underline{N} \, \underline{n} \cdot \underline{e}_3 = N_{3n} = Q_n \right\} \text{tagliante}$$

$$\underline{m} \cdot \underline{n} = \underline{e}_3 \times \underline{M} \, \underline{n} \cdot \underline{n} = \underline{M} \, \underline{n} \cdot (-\underline{\tau}) = -M_{\tau n}$$

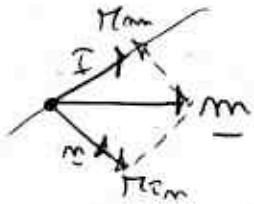


85 $$\underline{m} \cdot \underline{\tau} = \underline{e}_3 \times \underline{M} \, \underline{n} \cdot \underline{\tau} = \underline{M} \, \underline{n} \cdot (\underline{n}) = M_{nn}$$

(mom. torcente e mom. flettente)



- M_{nn}: fibra ruota intorno a τ
- $M_{\tau n}$: fibra ruota intorno a n



Vogliamo prob. 2D sup. in sup. media:

$\ulcorner$ $\mathrm{Div}\, \underline{v} = \underline{v}_{,i} \cdot \underline{e}_i$; se $\underline{v} = f$ di 2 variabili,

- $^{s}\mathrm{Div}\, \underline{v} \quad \underline{v}_{,\alpha} \cdot \underline{e}_\alpha = \underline{v}_{,1} \cdot \underline{e}_1 + \underline{v}_{,2} \cdot \underline{e}_2$

↳ "diverg. superficiale"

- $\nabla \varphi = \varphi_{,i}\, \underline{e}_i \rightarrow {}^{s}\nabla \varphi = \varphi_{,\alpha}\, \underline{e}_\alpha$
- ${}^{s}\nabla \underline{v} = \underline{v}_{,\alpha} \otimes \underline{e}_\alpha$
- ${}^{s}\mathrm{Div}\, \underline{A} = \underline{A}_{,\alpha}\, \underline{e}_\alpha$

${}^{s}\mathrm{Div}\,(\underline{A}^T \underline{v}) = (\underline{A}^T \underline{v})_{,\alpha} \cdot \underline{e}_\alpha = (\underline{A}^T_{,\alpha}\, \underline{v} + \underline{A}^T \underline{v}_{,\alpha}) \cdot \underline{e}_\alpha =$

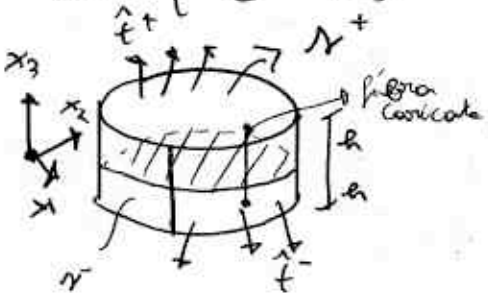
$= \underline{v} \cdot \underline{A}_{,\alpha}\, \underline{e}_\alpha + \underline{v}_{,\alpha} \cdot \underline{A}\, \underline{e}_\alpha = \underline{v} \cdot {}^{s}\mathrm{Div}\, \underline{A} +$

$+ \underline{A} \cdot (\underline{v}_{,\alpha} \otimes \underline{e}_\alpha) = \underline{v} \cdot {}^{s}\mathrm{Div}\, \underline{A} + \underline{A} \cdot {}^{s}\nabla \underline{v}$

È la vers. superficiale della nota identità diff.:

$\mathrm{div}(\underline{T}^T \underline{v}) = \mathrm{div}\, \underline{T} \cdot \underline{v} + \underline{T} \cdot \nabla \underline{v}$

${}^{s}\mathrm{div}(\underline{A}^T \underline{v}) = {}^{s}\mathrm{div}\, \underline{A} \cdot \underline{v} + \underline{A} \cdot {}^{s}\nabla \underline{v}$ $\lrcorner$



ν media a $x_3 = 0$.

Quali cariche possono agire?

$\hat{t}$: tensioni assegnate $\hat{t}^+$ o $\hat{t}^-$

Sul mantello spost. assegn. o tens.

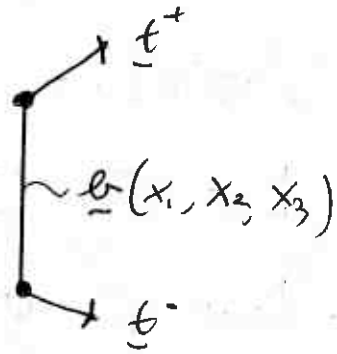
Se abbiamo fibra con tens. assegnate, $\hat{\underline{f}} = \int_{-h}^{h} \hat{\underline{t}}\, dx_3$

$$\hat{\underline{C}} = \int_{-h}^{h} x_3\, \hat{\underline{t}}\, dx_3$$

Cons. fibra caricata:

$$\underline{q} = \int_{-h}^{h} \underline{b}\, dx_3 + \hat{\underline{t}}^{+} + \hat{\underline{t}}^{-} \quad \left[F \cdot \frac{1}{L^2}\right] \text{ su } r$$

forza per unità di sup.



$$\underline{d} = \int_{-h}^{h} \underline{b}\, x_3\, dx_3 + h\left(\hat{\underline{t}}^{+} + \hat{\underline{t}}^{-}\right)$$: coppia per unità di sup

Carichi esterni riscritti a $\underline{q}$ e $\underline{d}$ sul piano medio.

Se piastra è in equil., $\operatorname{div} \underline{T} + \underline{b} = 0$ in P.

Int. su spessore:

$$\underline{T}_{,i}\, \underline{e}_i = \underline{T}_{,1}\, \underline{e}_1 + \underline{T}_{,2}\, \underline{e}_2 + \underline{T}_{,3}\, \underline{e}_3$$

$$\operatorname{div} \underline{T} = \underline{T}_{,\alpha}\, \underline{e}_\alpha + \underline{T}_{,3}\, \underline{e}_3$$

$$\int_{-h}^{h} \left(\underline{T}_{,\alpha}\, \underline{e}_\alpha + \underline{T}_{,3}\, \underline{e}_3 + \underline{b}\right) dx_3 = \underline{0}$$

$$\frac{\partial}{\partial x_\alpha}\left(\int_{-h}^{h} \underline{T}\, dx_3\right) \underline{e}_\alpha + \left[\underline{T}\, \underline{e}_3\right]_{-h}^{h} + \int_{-h}^{h} \underline{b}\, dx_3 = \underline{0}$$

$$\left[\int_{-h}^{h} \underline{T}_{,3}\, \underline{e}_3\, dx_3 = \int_{-h}^{h} \left(\underline{T}\, \underline{e}_3\right)_{,3} dx_3 = \left[\underline{T}\, \underline{e}_3\right]_{-h}^{h}\right]^{*}$$

Su s^+: $\underline{T}\, \underline{n} = \hat{\underline{t}}^{+}$ ovvero $\underline{T}\, \underline{e}_3 = \hat{\underline{t}}^{+}$

Su s^-: $\underline{T}\, \underline{n} = \hat{\underline{t}}^{-}$ " $-\underline{T}\, \underline{e}_3 = \hat{\underline{t}}^{-}$ (normale sempre verso esterno del corpo)

Sia ∂: $= \underline{T}\,\underline{e}_3\Big|_{x_3=h} - \underline{T}\,\underline{e}_3\Big|_{x_3=-h} = \hat{\underline{t}}^+ + \hat{\underline{t}}^-$

Sostituisco:

$$\underbrace{\frac{\partial}{\partial x_\alpha}\left(\int_{-h}^{h} \underline{T}\, dx_3\right)\underline{e}_\alpha}_{\underline{N}_{,\alpha}\,\underline{e}_\alpha} + \underbrace{\left(\hat{\underline{t}}^+ + \hat{\underline{t}}^- + \int_{-h}^{h} \underline{b}\, dx_3\right)}_{\underline{q}} = \underline{0}$$

$$\boxed{\operatorname{div} \underline{N} + \underline{q} = \underline{0}} \rightarrow \text{C.V.D.}$$

$$\int_{-h}^{h} x_3 \left(\operatorname{div} \underline{T} + \underline{b}\right) = \underline{0}$$

$$\int_{-h}^{h} x_3 \left(\underline{T}_{,\alpha}\,\underline{e}_\alpha + \underline{T}_{,3}\,\underline{e}_3 + \underline{b}\right) = \underline{0} \quad \left[x_3 \text{ è cost. risp. a } x_1, x_2\right]$$

$$\left[\int_{-h}^{h} \left((\underline{T} x_3)_{,\alpha}\,\underline{e}_\alpha\right)\right] \quad \left[\int_{-h}^{h} x_3\, \underline{T}_{,3}\,\underline{e}_3 = \int_{-h}^{h} \left(x_3\, \underline{T}\,\underline{e}_3\right)_{,3} - \int_{-h}^{h} \underline{T}\,\underline{e}_3\right]$$

Quindi:

$$\int_{-h}^{h} \left((\underline{T} x_3)_{,\alpha}\,\underline{e}_\alpha + \underbrace{(x_3\, \underline{T}\,\underline{e}_3)_{,3}}_{①} - \underline{T}\,\underline{e}_3 + \underbrace{x_3\, \underline{b}}_{②}\right) dx_3 = 0$$

① : $\left[x_3\, \underline{T}\,\underline{e}_3\right]_{-h}^{h} = h\,\hat{\underline{t}}^+ - \left(x_3\, \underline{T}\,\underline{e}_3\right)_{x_3=-h} =$

$= h\,\hat{t}^+ - (-h)(-\hat{\underline{t}}^-) = h\left(\hat{\underline{t}}^+ - \hat{\underline{t}}^-\right)$

① + ② = $\underline{0}$! Allora:

$$\int_{-h}^{h} \left(\left(\underline{T}\, x_3 \right)_{,\alpha} \cdot \underline{e}_\alpha - \underline{T}\, \underline{e}_3 \right) d x_3 + \underline{d} = \underline{0}$$

$$\frac{\partial}{\partial x_\alpha} \left(\int_{-h}^{h} x_3 \underline{T} \right) \underline{e}_\alpha - \int_{-h}^{h} \underline{T}\, \underline{e}_3\, d x_3 + \underline{d} = \underline{0}$$

$$\underline{\Pi}_{,\alpha}\, \underline{e}_\alpha - \underline{N}\, \underline{e}_3 + \underline{d} = 0 \quad \text{ovvero}$$

$$\boxed{\operatorname{div} \underline{\Pi} - \underline{N}\, \underline{e}_3 + \underline{d} = \underline{0}}$$

Ma queste 2 eq. non sono sufficienti. [eq. GENERALI]
Occorre esprimere risult. in termini di comp. di spost. per risolvere equilibrio.

21/4/2009

$$\underline{N} = \int_{-h}^{h} \underline{T}\, d x_3 \quad ; \quad \underline{\Pi} = \int_{-h}^{h} \underline{T}\, x_3\, d x_3$$

$$N_{i,\alpha} = \int_{-h}^{h} T_{i\alpha}\, d x_3 \quad ; \quad \Pi_{\alpha\beta} = \int_{-h}^{h} T_{\alpha\beta}\, x_3\, d x_3$$

Dobbiamo usare le <u>EQUAZIONI COSTITUTIVE</u> per esprimere

$$\underline{T} = \mathbb{C}[\underline{E}] \quad ; \quad T_{is} = \mathbb{C}_{isnl}\, E_{nl}$$

Materiale isotropo ha 2 comp. indip.:

$$\begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1122} & 0 & 0 & 0 \\ & \mathbb{C}_{1111} & \mathbb{C}_{1122} & 0 & 0 & 0 \\ & & \mathbb{C}_{1111} & 0 & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & & & \mathbb{C}_{1212} & 0 & \\ & & & & & \mathbb{C}_{1212} \end{bmatrix}$$

$$\mathbb{C}_{1212} = \left(\mathbb{C}_{1111} - \mathbb{C}_{1122}\right)/2$$

$$\begin{cases} \mathbb{C}_{1111} = 2\mu + \lambda \\ \mathbb{C}_{1122} = \lambda \end{cases} \quad \Bigg| \quad \mathbb{C}_{1212} = \mu$$

Eq. di Lamé:

$$\underline{T} = 2\mu \underline{E} + \lambda (\mathrm{tr}\, \underline{E}) \underline{I}$$

$$T_{ij} = 2\mu E_{ij} + \lambda E_{hh} \delta_{ij}$$

Mat. trasv. isotropo: comport. = in dir ⊥ a dir data: (ex (3))

direzione di isotropia trasversa assegnata

$$\begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & 0 & 0 & 0 \\ & \mathbb{C}_{1111} & \mathbb{C}_{1133} & 0 & 0 & 0 \\ & & \mathbb{C}_{3333} & 0 & 0 & 0 \\ & & & \mathbb{C}_{1212} & 0 & 0 \\ & & & & \mathbb{C}_{2323} & 0 \\ & & & & & \mathbb{C}_{2323} \end{bmatrix}$$

da 2 a 5

$$\mathbb{C}_{1111} = 2\mu + \lambda \qquad \mathbb{C}_{1133} = \tilde{\lambda} \qquad \mathbb{C}_{1212} = (\mathbb{C}_{1111} - \mathbb{C}_{1122})/2 = \mu$$

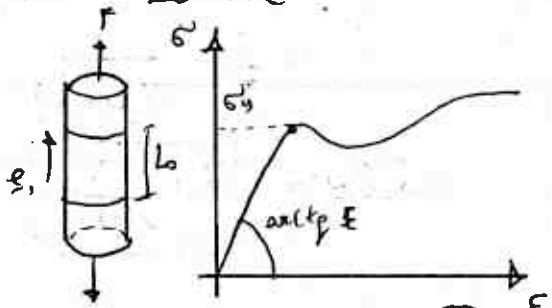
$$\mathbb{C}_{1122} = \lambda \qquad \mathbb{C}_{3333} = \hat{\lambda} \qquad \mathbb{C}_{2323} = \hat{\mu}$$

Ma non si usano i moduli di Lamé

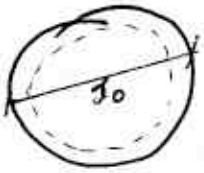
$$\underline{T} = 2\mu \underline{E} + \lambda (\mathrm{tr}\, \underline{E}) \underline{I}$$

Si usano E e ν, significato fisico più chiaro. Nella prova qui ho T_{11} $\quad T_{22} = T_{33} = 0$

$E = T_{11} / E_{11}$;



$\sigma = \frac{F}{A_0}$; $\varepsilon = \frac{L - L_0}{L_0}$



$$\varepsilon_{22} = \frac{d - d_0}{d_0}$$ (contrazione del provino)

$$\varepsilon_{22} = -\frac{\nu}{E} T_{11} \quad ; \quad \varepsilon_{11} = \frac{T_{11}}{E} \quad ; \quad T_{12} = 2G\,\varepsilon_{12}$$

Comp. diagonali sono la metà dei scorrimenti: $\tau_{12} = G\gamma_{12}$

[sempre mat. isotropo]:

$$T_{11} = 2\mu\,\varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$
$$T_{22} = 2\mu\,\varepsilon_{22} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$
$$T_{33} = 2\mu\,\varepsilon_{33} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$
$$T_{12} = 2\mu\,\varepsilon_{12}$$
$$T_{23} = 2\mu\,\varepsilon_{23}$$
$$T_{31} = 2\mu\,\varepsilon_{31}$$

[Espressione moduli tecnici in f. mod. sé lame']

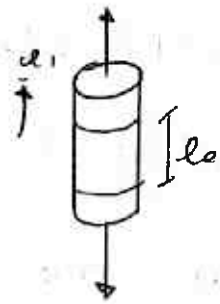
$$(T_{11} + T_{22} + T_{33}) = (2\mu + 3\lambda)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

$$T_{11} = 2\mu\,\varepsilon_{11} + \frac{\lambda}{2\mu + 3\lambda}(T_{11} + T_{22} + T_{33})$$

$$\varepsilon_{11} = \frac{1}{2\mu} T_{11} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}(T_{11} + T_{22} + T_{33})$$

$$\varepsilon_{11} = \frac{2(\mu+\lambda)}{2\mu(2\mu+3\lambda)} T_{11} - \frac{2(\mu+\lambda)}{2\mu(2\mu+3\lambda)} \frac{\lambda}{2(\mu+\lambda)}(T_{22} + T_{33})$$

$$\boxed{E = \frac{2\mu(2\mu+3\lambda)}{2(\mu+\lambda)}}$$



$$\varepsilon_{22} = \text{(//)}\; T_{22} - \text{(//)}\,(T_{11} + T_{33})$$

$$-\frac{1}{E}\frac{\lambda}{2(\mu+\lambda)} T_{11}$$; lo vogliamo come $-\frac{\nu}{E} T_{11}$

e allora $\boxed{\nu = \frac{\lambda}{2(\mu+\lambda)}}$

(9) $C_{1212} = \frac{C_{1111} - C_{1122}}{2}$ ⇝ rel. tra E e G

$$\varepsilon_{11} = \frac{1}{E}\left(T_{11} - \nu\left(T_{22} + T_{33}\right)\right) \qquad \varepsilon_{12} = \frac{1}{2G} T_{12}$$

$$\varepsilon_{22} = \frac{1}{E}\left(T_{22} - \nu\left(T_{11} + T_{33}\right)\right)$$

$$\varepsilon_{33} = \frac{1}{E}\left(T_{33} - \nu\left(T_{22} + T_{11}\right)\right)$$

Invertendo:

$$T_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{11} + \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_{22} + \varepsilon_{33}\right)$$

[Dato $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ e $\mu = \frac{E}{1+\nu}$ se sostituite in Lamé, da queste espressione.]

Mat. <u>trasversv. isotropo</u>

$$T_{11} = 2\mu\,\varepsilon_{11} + \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) + \tilde{\lambda}\,\varepsilon_{33}$$

$$T_{22} = 2\mu\,\varepsilon_{22} + \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) + \tilde{\lambda}\,\varepsilon_{33}$$

$$T_{33} = \hat{\lambda}\,\varepsilon_{33} + \tilde{\lambda}\left(\varepsilon_{11} + \varepsilon_{22}\right)$$

$$\left.\begin{aligned} T_{12} &= 2\mu\,\varepsilon_{12} \\ T_{13} &= 2\hat{\mu}\,\varepsilon_{13} \\ T_{23} &= 2\hat{\mu}\,\varepsilon_{23} \end{aligned}\right\} \text{si invertono immediat.}$$

Vogliamo eq. (analoghe x mat. is.) con E, G, ν.

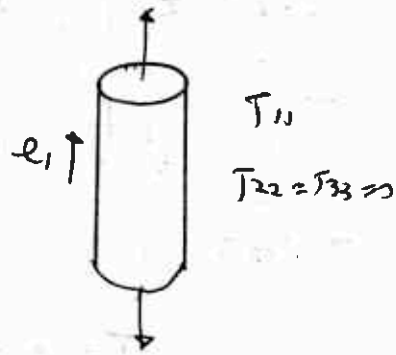
Cons. deformazioni con $\varepsilon_{33} = 0$ [caso + comune]

$$\left(T_{11} + T_{22}\right) = 2\left(\mu + \lambda\right)\left(\varepsilon_{11} + \varepsilon_{22}\right)$$

$$T_{11} = 2\mu \varepsilon_{11} + \frac{\lambda}{2(\mu+\lambda)} (T_{11} + T_{22})$$

$$\varepsilon_{11} = \frac{2\mu+\lambda}{4\mu(\mu+\lambda)} T_{11} - \frac{2\mu+\lambda}{4\mu(\mu+\lambda)} \frac{\lambda}{2\mu+\lambda} T_{22}$$

$$\boxed{E = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}} \quad ; \quad \boxed{\nu = \frac{\lambda}{2\mu+\lambda}}$$



$$\Downarrow$$

$$\lambda = \frac{E\nu}{1-\nu^2} \quad ; \quad 2\mu = \frac{E}{1+\nu}$$

$$2\mu + \lambda = \frac{E}{1-\nu^2}$$

Sostit:

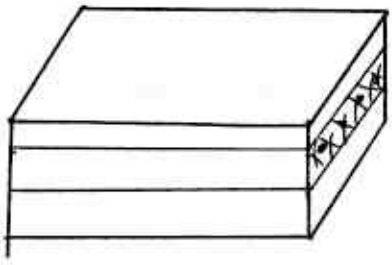
$$\boxed{\begin{aligned} T_{11} &= \frac{E}{1-\nu^2}\left(\varepsilon_{11} + \nu \varepsilon_{22}\right) \\ T_{22} &= \frac{E}{1-\nu^2}\left(\varepsilon_{22} + \nu \varepsilon_{11}\right) \\ T_{12} &= \underbrace{2G}_{\frac{E}{1+\nu}} \varepsilon_{12} \; ; \; T_{\alpha 3} = 2\hat{G}\varepsilon_{\alpha 3} \end{aligned}}$$

→ forma di rel. costitutive che sarà usata

Mat. <u>ortotropo (romb.)</u>:

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & & & & C_{2323} & 0 \\ & & & & & C_{3131} \end{bmatrix}$$

Oggi si studiano mat. compositi



Singolo strato ha ottima res. longit., res. trasver. nulla. Fibre quindi orientate solitary.

$\forall$ strato conv queste eq. cost.:

$$T_{11} = C_{1111}\,\varepsilon_{11} + C_{1122}\,\varepsilon_{22} + C_{1133}\,\varepsilon_{33}$$

$$T_{22} = C_{2211}\,\varepsilon_{11} + C_{2222}\,\varepsilon_{22} + C_{2233}\,\varepsilon_{33}$$

$$T_{33} = C_{3311}\,\varepsilon_{11} + C_{3322}\,\varepsilon_{22} + C_{3333}\,\varepsilon_{33}$$

$$T_{12} = 2C_{1212}\,\varepsilon_{12} \quad (= C_{1212}\,\varepsilon_{12} + C_{1221}\,\varepsilon_{21})$$

$$T_{23} = 2C_{2323}\,\varepsilon_{23}$$

$$T_{31} = 2C_{3131}\,\varepsilon_{31}$$

$$\varepsilon_{11} = \frac{1}{\Delta}\begin{vmatrix} T_{11} & C_{1122} & C_{1133} \\ T_{22} & C_{2222} & C_{2233} \\ T_{33} & C_{3322} & C_{3333} \end{vmatrix}; \quad \Delta = \det\begin{vmatrix} C_{1111} & C_{1122} & C_{1133} \\ & C_{2222} & C_{2233} \\ \text{sym} & & C_{3333} \end{vmatrix}$$

(analogo per ε_{22} ed ε_{33})

$$\varepsilon_{11} = \frac{1}{\Delta}\Big(T_{11}\left(C_{2222}C_{3333} - C_{2233}C_{3322}\right) + T_{22}\left(C_{1133}C_{3322} + - C_{1122}C_{3333}\right) + T_{33}\left(C_{1122}C_{2233} - C_{1133}C_{2222}\right)\Big)$$

$$\varepsilon_{11} = \frac{1}{E_{(1)}}\left(T_{11} - \nu_{(12)}T_{22} - \nu_{(13)}T_{33}\right)$$

$$\varepsilon_{22} = \frac{1}{E_{(2)}}\left(T_{22} - \nu_{(21)}T_{11} - \nu_{(23)}T_{33}\right)$$

$$\varepsilon_{33} = \frac{1}{E_{(3)}}\left(T_{33} - \nu_{(31)} T_{11} - \nu_{(32)} T_{22}\right)$$

Conv. $\varepsilon_{33} = 0$. Hp:

$$T_{11} = C_{1111}\,\varepsilon_{11} + C_{1122}\,\varepsilon_{22}$$

$$T_{22} = C_{2211}\,\varepsilon_{11} + C_{2222}\,\varepsilon_{22}$$

$$\varepsilon_{11} = \frac{1}{\Delta'}\begin{bmatrix} T_{11} & C_{1122} \\ T_{22} & C_{2222} \end{bmatrix} = \frac{1}{E_{(1)}}\left(T_{11} - \nu_{(12)} T_{22}\right)$$

$$\varepsilon_{22} = \frac{1}{\Delta'}\begin{bmatrix} C_{1111} & T_{11} \\ C_{2211} & T_{22} \end{bmatrix} = \frac{1}{E_{(2)}}\left(T_{22} - \nu_{(21)} T_{11}\right)$$

Quindi $\underline{\nu_{(12)} E_{(2)} = \nu_{(21)} E_{(1)}}$ Invert:

$$\boxed{\begin{aligned} T_{11} &= \frac{E_{(1)}}{1 - \nu_{(12)}\nu_{(21)}}\left(\varepsilon_{11} + \nu_{(21)}\,\varepsilon_{22}\right) \\ T_{22} &= \frac{E_{(2)}}{1 - \nu_{(12)}\nu_{(21)}}\left(\varepsilon_{22} + \nu_{(12)}\,\varepsilon_{11}\right) \\ T_{12} &= 2G_{(12)}\,\varepsilon_{12}\ ;\ T_{23} = 2G_{(23)}\,\varepsilon_{23}\ ;\ T_{31} = 2G_{(31)}\,\varepsilon_{31} \end{aligned}}$$

Mat. <u>ortotropo (tetragonale)</u>:

Semplific.: $C_{2222} = C_{1111}$; $C_{2233} = C_{1133}$; $C_{2323} = C_{3131}$

[Come isotropia in dir $\perp$ che sono uguali]

$$\boxed{\begin{aligned} &E_{(1)} = E_{(2)}\ ;\ \nu_{(12)} = \nu_{(21)} \\ &G_{(23)} = G_{(31)} \end{aligned}}$$

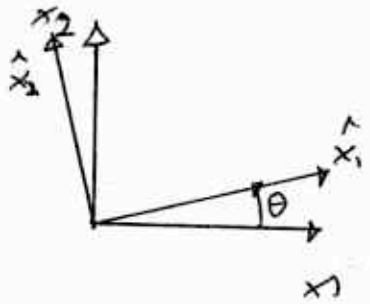
Useremo

$$T_{11} = \frac{E}{1-\nu^2}\left(\varepsilon_{11} + \nu\,\varepsilon_{22}\right)\ ;\ T_{12} = \frac{E}{1+\nu}\,\varepsilon_{12}$$

$$T_{22} = \frac{E}{1-\nu^2}\left(\varepsilon_{22} + \nu\,\varepsilon_{11}\right)\ ;\ T_{23} = 2\hat{\mu}\,\varepsilon_{23}$$

CAMBIAMENTO DI COORDINATE

Se abbiamo piastre con lamine ortotrope non disposte a 90°, prima abbiamo conv. e_1 ed e_2 come dir, ma se c'è inclinazione θ?



Sistema originario $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$

" secondo $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$

$$\underline{Q} = \begin{bmatrix} (\hat{\underline{e}}_1)_1 & (\hat{\underline{e}}_2)_1 & (\hat{\underline{e}}_3)_1 \\ (\hat{\underline{e}}_1)_2 & (\hat{\underline{e}}_2)_2 & (\hat{\underline{e}}_3)_2 \\ (\hat{\underline{e}}_1)_3 & (\hat{\underline{e}}_2)_3 & (\hat{\underline{e}}_3)_3 \end{bmatrix}$$

$Q_{12} = \hat{\underline{e}}_1 \cdot \underline{e}_2$. Se vettore ha $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ nel 1° sist. e ha $\hat{\underline{u}} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix}$ nel 2° sist., $\underline{u} = \underline{Q}\,\hat{\underline{u}}$

Per un tensore $\underline{A} = \underline{Q}\,\hat{\underline{A}}\,\underline{Q}^T$

Se $\underline{u} = \underline{A}\,\underline{v}$, $\hat{\underline{u}} = \hat{\underline{A}}\,\hat{\underline{v}}$, ho stessa legge.

$\underline{T} = \underline{Q}\,\hat{\underline{T}}\,\underline{Q}^T$; $\underline{T} = \mathbb{C}[\underline{E}]$ e $\hat{\underline{T}} = \hat{\mathbb{C}}[\hat{\underline{E}}]$ (rel. tra componenti, matr. di componenti)

$$\mathbb{C}[\underline{E}] = \underline{Q}\,\hat{\mathbb{C}}\left[\underline{Q}^T \underline{E}\,\underline{Q}\right]\underline{Q}^T$$

$$\left[\underline{E} = \underline{Q}\,\hat{\underline{E}}\,\underline{Q}^T;\ \hat{\underline{E}} = \underline{Q}^T \underline{E}\,\underline{Q}\right]$$

$$\mathbb{C}_{ijkl} = \mathbb{C}\left[\underline{e}_k \otimes \underline{e}_l\right] \cdot \left(\underline{e}_i \otimes \underline{e}_j\right) =$$

$$= \underline{Q}\,\hat{\mathbb{C}}\left[\underbrace{\underline{Q}^T\left(\underline{e}_k \otimes \underline{e}_l\right)\underline{Q}}_{\text{visto più nei richiami}}\right]\underline{Q}^T \cdot \left(\underline{e}_i \otimes \underline{e}_j\right) =$$

$$= \hat{\mathbb{C}}\left[\underline{Q}^T \underline{e}_k \otimes \underline{Q}^T \underline{e}_l\right] \cdot \left(\underline{Q}^T \underline{e}_i \otimes \underline{Q}^T \underline{e}_j\right)$$

$$\underline{Q}^T \underline{e}_k = (Q^T)_{ik}\, \hat{\underline{e}}_i = Q_{ki}\, \hat{\underline{e}}_i$$

$$\underline{Q}^T \underline{e}_i = Q_{im}\, \hat{\underline{e}}_m \qquad \underline{Q}^T \underline{e}_l = Q\, \underline{e}_q\, \hat{\underline{e}}_q$$

$$\underline{Q}^T \underline{e}_j = Q_{jn}\, \hat{\underline{e}}_n$$

$$\underline{Q}^T \underline{e}_k = Q_{kp}\, \hat{\underline{e}}_p$$

$$C_{ijkl} = Q_{im}\, Q_{jn}\, Q_{kp}\, Q_{lq}\, \hat{C}_{mnpq}$$

Qui $\underline{e}_3 = \hat{\underline{e}}_3$.

$$\underline{Q} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_{1111} = \hat{C}_{1111} \cos^4\theta + 2\left(\hat{C}_{1122} + 2\hat{C}_{1212}\right)\cos^2\theta \sin^2\theta + \hat{C}_{2222} \sin^4\theta \quad (\text{etc})$$

Alcune comp. prima nulle ora non lo sono:

$$C_{1112} = \left(\hat{C}_{1111} - \hat{C}_{1122} - 2\hat{C}_{2222}\right)\cos^3\theta \sin\theta + \left(\hat{C}_{1122} + 2\hat{C}_{1212} - \hat{C}_{2222}\right)\cos\theta \sin^3\theta$$

(!)

Si generano elasticità "apparenti"

Se $\theta = \frac{\pi}{2}$? $C_{1111} = \hat{C}_{2222}$, $C_{2233} = \hat{C}_{1133}$,

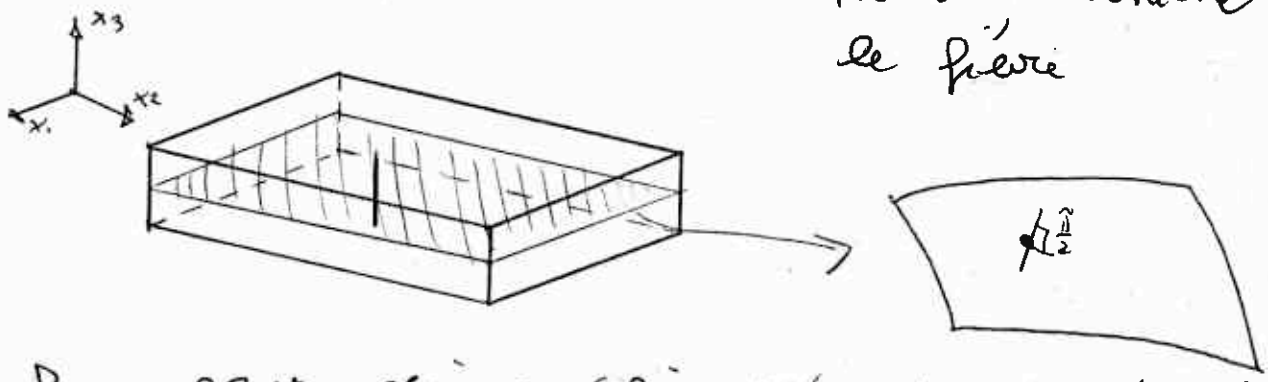
$C_{3333} = \hat{C}_{3333}$

27/4/09

(→)

TEORIA DI KIRCHHOFF-LOVE (o TEORIA CLASSICA DELLE PIASTRE)

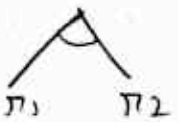
Piastra uniforme di tutte le fibre



Per effetto dei carichi piano medio si deforma. A deform. si conservano RETTILINEI, non subiscono ESTENSIONI e rimangono $\perp$ a piano medio.

Cosa vuol dire in termini meccanici:

$\varepsilon_{33} = 0$. Lo scorrimento γ_{12} di $\pi_1, \pi_2 \parallel$ a x_1, x_2 è $\gamma_{12} = \frac{\pi}{2} - \theta$



Se fibre rimangono $\perp$, angolo tra fibra e segmenti di x_1, x_2 deformati rimane uguale!

$\varepsilon_{31} = \varepsilon_{32} = 0$. Vale $\forall$ punto, quindi le ipot. $\equiv$ a

$$\begin{cases} \varepsilon_{33} = 0 \\ \varepsilon_{31} = 0 \\ \varepsilon_{32} = 0 \end{cases}$$

Cond. differenz. nel campo di spost. che integriamo. Da queste si ha:

$$\begin{cases} u_{3,3} = 0 \Rightarrow u_3 = w(x_1, x_2) \\ \frac{1}{2}(u_{3,1} + u_{1,3}) = 0 \Rightarrow u_{3,1} + u_{1,3} = 0 \\ u_{3,2} + u_{2,3} = 0 \end{cases}$$

Sost. nella 2: $u_{1,3} = -u_{3,1} = -w_{,1}$

quindi $u_{1,3} = -w_{,1}$. Integro su 3 e $u_1 = \tilde{u}_1(x_1, x_2) - x_3 w_{,1}(x_1, x_2)$

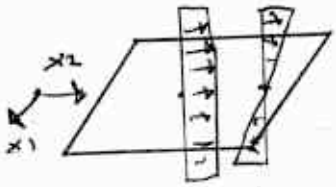
Idem per ③: $u_2 = \tilde{u}_2(x_1, x_2) - x_3 w_{,2}(x_1, x_2)$

$$u_1 = \tilde{u}_1(x_1, x_2) - x_3 w_{,1}(x_1, x_2)$$

$$u_2 = \tilde{u}_2(x_1, x_2) - x_3 w_{,2}(x_1, x_2)$$

$u_3 = w(x_1, x_2)$

Tutti i punti hanno stesso spost. nella direz. 3



Quale def. corrim.?

$$\underline{u} = (\tilde{u}_\alpha - x_3 w_{,\alpha})\, \underline{e}_\alpha + w\, \underline{e}_3$$

$$\nabla \underline{u} = (\tilde{u}_\alpha - x_3 w_{,\alpha})_{,\beta}\, \underline{e}_\alpha \otimes \underline{e}_\beta + w_{,\alpha}\, \underline{e}_3 \otimes \underline{e}_\alpha + - w_{,\alpha}\, \underline{e}_\alpha \otimes \underline{e}_3$$

[infatti: $\nabla \underline{v} = \underline{v}_{,i} \otimes \underline{e}_i = \underline{v}_{,\alpha} \otimes \underline{e}_\alpha + \underline{v}_{,3} \otimes \underline{e}_3$]

$$\nabla \underline{u}^T = (\tilde{u}_\alpha - x_3 w_{,\alpha})_{,\beta}\, \underline{e}_\beta \otimes \underline{e}_\alpha + w_{,\alpha}\, \underline{e}_\alpha \otimes \underline{e}_3 + - w_{,\alpha}\, \underline{e}_3 \otimes \underline{e}_\alpha$$

; essendo $\underline{\varepsilon} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$, si hai

$$\begin{cases} \varepsilon_{11} = \tilde{u}_{1,1} - x_3\, w_{,11} \\ \varepsilon_{22} = \tilde{u}_{2,2} - x_3\, w_{,22} \\ \varepsilon_{12} = \frac{1}{2}(\tilde{u}_{1,2} + \tilde{u}_{2,1}) - x_3\, w_{,12} \end{cases}$$

→ sono le comp. non nulle del tensore di deform.

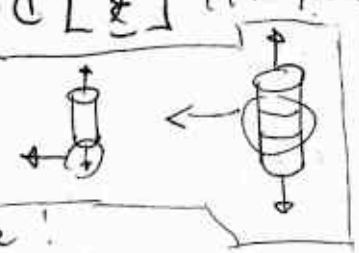
$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\tilde{u}_{\alpha,\beta} + \tilde{u}_{\beta,\alpha}) - x_3\, w_{,\alpha\beta} \quad [\alpha, \beta = 1,2]$$

Si è cons. ipotesi con deformazioni non libere, dette VINCOLI INTERNI ($\varepsilon_{33} = \varepsilon_{31} = \varepsilon_{32} = 0$).

Quando corpo è RIGIDO si assume che $\varepsilon_{ij} = 0$ ($i,j = 1,2,3$)

Nei corpi privi di vincoli interni, $\underline{T} = \mathbb{C}[\underline{\varepsilon}]$ (in quelli rigidi $\underline{T} = \underline{0}$). Si divide lo sforzo nella parte ATTIVA (det. da deform. eq.) e REATTIVA che ∝ dalla deformazione!

non può essere 0!



(99) $\underline{T} = \underline{T}^A + \underline{T}^R$

Quando si impongono limitaz. a $\underline{E}$, questo è sym in un suo sottoinsieme. Si definisce

$$D = \{ \underline{E} \in Sym \,/\, E_{3i} = 0,\ i = 1,2,3 \}$$

↳ spazio delle DEFORMAZIONI AMMISSIBILI

$$D^{\perp} = \{ S \in Sym \,/\, \underline{S} \cdot \underline{E} = 0,\ \forall \underline{E} \in D \}$$

Su T^R si fa l'ipotesi che non compongano lavoro [ipotesi fondamentale]

$$\underline{T}^R \cdot \underline{E} = 0,\ \forall \underline{E} \in D \quad [\forall \text{ deform ammissibile}].$$

Vale nei vincoli privi di attrito, non vogliamo produrre energia. Quindi $\underline{T}^R \in D^{\perp}$

In generale $\underline{T}^A$ può $\in Sym$ [che è diviso in $\underline{D}$ e $\underline{D}^{\perp}$].
Poiché $\underline{T} = \underline{T}^A + \underline{T}^R$ e $\underline{T}^R \in D^{\perp}$ indeterminata, la parte di $\underline{T}^A \in D^{\perp}$ si somma a q. indet. e quindi essa stessa è indet. Allora la trascuriamo e $\underline{T}^A \in D$.
Allora $\underline{T}^A = \mathbb{C}[\underline{E}]$, $\mathbb{C} : D \longrightarrow D$.

Qui $D^{\perp} = \text{span}\left\{ \underline{e}_3 \otimes \underline{e}_3,\ \frac{1}{\sqrt{2}}(\underline{e}_3 \otimes \underline{e}_\alpha + \underline{e}_\alpha \otimes \underline{e}_3) \right\}$

$$\underline{T}^A = T^A_{\alpha\beta}\, \underline{e}_\alpha \otimes \underline{e}_\beta\ ;\quad T^A_{11},\ T^A_{12},\ T^A_{21},\ T^A_{22}$$

$$\underline{T}^R = T_{3\alpha}(\underline{e}_\alpha \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_\alpha) + T^R_{33}\, \underline{e}_3 \otimes \underline{e}_3$$

E da chi è determ. $\underline{T}^R$? Dall'EQUILIBRIO!

$$D = \{ \underline{E} \in Sym \,/\, E_{3i} = 0,\ i = 1,2,3 \}$$

$\underline{T}^A = \mathbb{C}[\underline{E}]$, $\mathbb{C}: D \to D$

$\underline{T}^R \in D^\perp$

Cons. 2 possibilità di materiale x questa piastra.

Se in dir. 3 fibre hanno comp. particolare, non è isotropo. <u>Trasversalm. isotr.</u> sì. Però se $\mathbb{C}: D \to D$ sorgono altre ipotesi

$$\begin{cases} T_{11} = 2\mu E_{11} + \lambda(E_{11} + E_{22}) + \tilde{\lambda} E_{33} \\ T_{33} = \hat{\lambda} E_{33} + \hat{\lambda}(E_{11} + E_{22}) \\ T_{3\alpha} = 2\hat{\mu} E_{3\alpha} \end{cases} \Rightarrow \underline{\hat{\lambda} = \tilde{\lambda} = 0}$$

$$\underline{\hat{\mu} = 0}$$

[per pura coerenza matematica]

Quindi con queste ipotesi si ha

$$\begin{cases} T_{11} = 2\mu E_{11} + \lambda(E_{11} + E_{22}) \\ T_{22} = 2\mu E_{22} + \lambda(E_{11} + E_{22}) \\ T_{12} = 2\mu E_{12} \end{cases}$$

Se avessimo preso direttam. mat. isotropo saremo giunti a questa SENZA cons. su vincoli interni.

Preferiamo la forma ingegneristica:

$$\begin{cases} T_{11} = \dfrac{E}{1-\nu^2}(E_{11} + \nu E_{22}) \\ T_{22} = \dfrac{E}{1-\nu^2}(E_{22} + \nu E_{11}) \\ T_{12} = \dfrac{E}{1+\nu} E_{12} \end{cases}$$

Supp. materiale <u>ORTOTROPO</u>.

(bis) Ex: piastra con irrigidimenti nelle 2 dir. $\perp$

Com. ortotropo ROMBICO:

$$\begin{cases} T_{11} = \dfrac{E_{(1)}}{1-\nu_{(12)}\nu_{(21)}}\left(\varepsilon_{11} + \nu_{(21)}\,\varepsilon_{22}\right) \\ T_{22} = \dfrac{E_{(2)}}{1-\nu_{(12)}\nu_{(21)}}\left(\varepsilon_{22} + \nu_{(12)}\,\varepsilon_{11}\right) \\ T_{12} = 2G_{(12)}\,\varepsilon_{12} \end{cases}$$

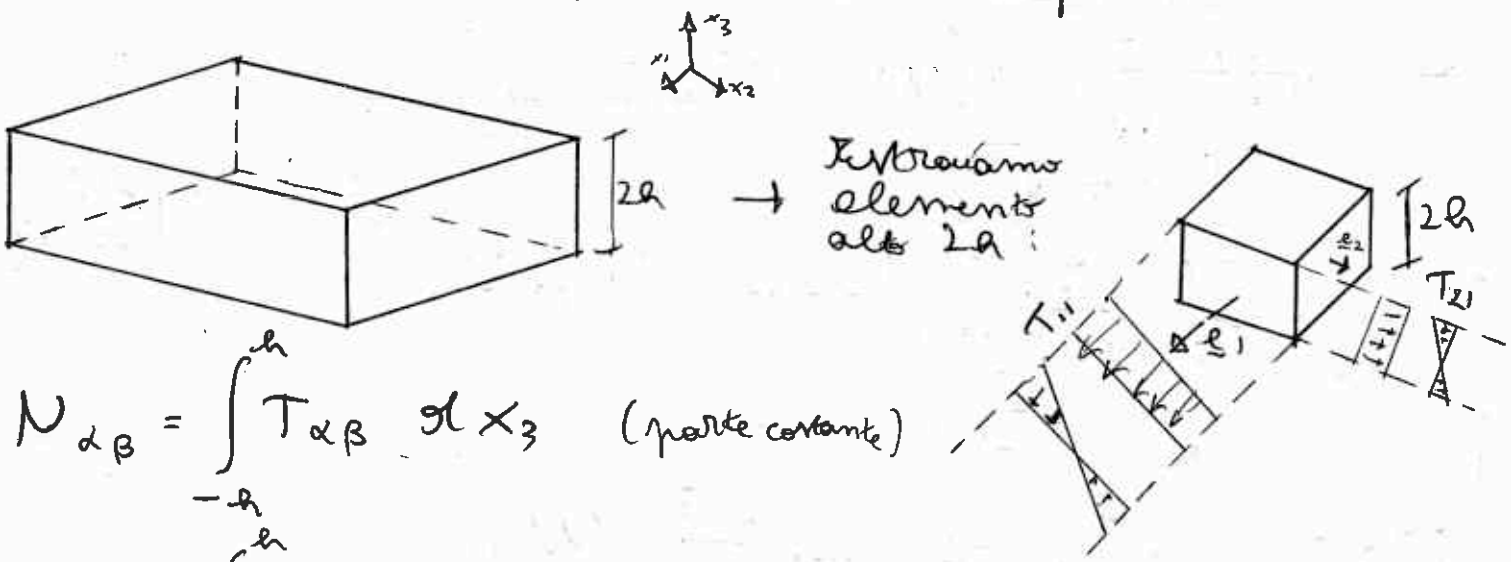
Per piastra trasv. isotropa:

$$T_{11} = \frac{E}{1-\nu^2}\left(\tilde{u}_{1,1} + \nu\,\tilde{u}_{2,2}\right) - \frac{E}{1-\nu^2}\left(w_{,11} + \nu\, w_{,22}\right)x_3$$

Per piastra ortotropa:

$$T_{11} = \frac{E_{(1)}}{1-\nu_{(12)}\nu_{(21)}}\left(\tilde{u}_{1,1} + \nu_{(21)}\,\tilde{u}_{2,2}\right) - \frac{E_{(1)}}{1-\nu_{(12)}\nu_{(21)}}\left(w_{,11} + \nu_{(21)}\, w_{,22}\right)x_3$$

Parte costante lungo l'altezza + parte variabile:



→ Estraiamo elemento alto $2h$:

$$N_{\alpha\beta} = \int_{-h}^{h} T_{\alpha\beta}\, dx_3 \quad \text{(parte costante)}$$

$$M_{\alpha\beta} = \int_{-h}^{h} x_3\, T_{\alpha\beta}\, dx_3 \quad \text{(parte variabile)}$$

Ex:

$$N_{11} = \int_{-h}^{h} T_{11}\, dx_3 = \frac{2hE}{1-\nu^2}\left(\tilde{u}_{1,1} + \nu\,\tilde{u}_{2,2}\right)$$

$$M_{11} = \int_{-h}^{h} x_3\, T_{11}\, dx_3 = \left[\int_{-h}^{h} x_3^2\, dx_3 = \left[\frac{x_3^3}{3}\right]_{-h}^{h} = \frac{2}{3}h^3\right]$$

$$= -\frac{2}{3} h^3 \frac{E}{1-\nu^2}\left(w_{,11} + \nu w_{,22}\right)$$

$$N_{22} = \int_{-h}^{h} T_{22}\, dx_3 = \frac{2hE}{1-\nu^2}\left(\tilde{u}_{3,2} + \nu \tilde{u}_{1,1}\right)$$

$$N_{12} = \int_{-h}^{h} T_{12}\, dx_3 = \frac{hE}{1+\nu}\left(\tilde{u}_{1,2} + \tilde{u}_{3,1}\right)$$

Poniamo $\boxed{D = \frac{2h^3}{3}\frac{E}{1-\nu^2}}$ → RIGIDEZZA FLESSIONALE per unità di lung. della piastra

Allora:

[mat. trasv. isotr.]

$$\begin{cases} M_{11} = -D\left(w_{,11} + \nu w_{,22}\right) \\ M_{22} = -D\left(w_{,22} + \nu w_{,11}\right) \\ M_{12} = \int_{-h}^{h} x_3 \frac{E}{1+\nu} T_{12} = \int_{-h}^{h} x_3 \frac{E(1-\nu)}{(1-\nu^2)} T_{12} = -D(1-\nu) w_{,12} \end{cases}$$

$$\begin{cases} M_{11} = -\underbrace{\frac{2h^3}{3}\frac{E_{(1)}}{1-\nu_{(12)}\nu_{(21)}}}_{D_{(1)}}\left(w_{,11} + \nu_{(21)} w_{,22}\right) \\ M_{22} = -\underbrace{\frac{2h^3}{3}\frac{E_{(2)}}{1-\nu_{(12)}\nu_{(21)}}}_{D_{(2)}}\left(w_{,22} + \nu_{(12)} w_{,11}\right) \\ M_{12} = \int_{-h}^{h} x_3 T_{12} = -2 G_{(12)} \frac{2h^3}{3} w_{,12} \end{cases}$$

[mat. ortotropo]

Ricorda: [sforzi membr.] $N_{\alpha\beta}$ dipende solo da $\tilde{u}_{\alpha,\beta}$

$M_{\alpha\beta}$ dipende " " $w_{,\alpha\beta}$

E i TAGLI? è PRESTAZIONE

(103) $Q_1 = N_{31} = \int_{-h}^{h} T_{31}\, dx_3$ ↳ lo ricaviamo dall'equil.

tridimensionale: $\operatorname{div} \underline{T} + \underline{b} = 0$; $T_{11,1} + T_{12,2} + T_{13,3} + b_1 = 0$
(dalla 2ª T_{23} e T_{33} dalla 3ª)
Ma ricordo: quando abbiamo fatto $\int_{-h}^{h} x_3 (\operatorname{div} \underline{T} + \underline{b}) = \underline{0}$
si è ottenuto:

$$M_{\alpha\beta,\beta} - Q_\alpha + d_\alpha = 0 \quad , \quad \alpha = 1,2$$

$$\left[M_{11,1} + M_{12,2} + d_1 = 0 ; \quad M_{21,1} + M_{22,2} - Q_2 + d_2 = 0 \right]$$

Però non sapendo T_{31}, esprimiamo Q_1 e Q_2 in f. dei carichi e M:

$$\boxed{Q_\alpha = M_{\alpha\beta,\beta} + d_\alpha} \quad \alpha = 1,2$$

EQUAZIONI DI EQUILIBRIO PIASTRA

Ricaviamo eq. equil. Scriviamo P.L.V. assumendo che f virtuale abbia stessa forma di quello ammissibile.

$$\underline{u} = (\tilde{u}_\alpha - x_3 w_{,\alpha}) \underline{e}_\alpha + w \underline{e}_3$$

allora

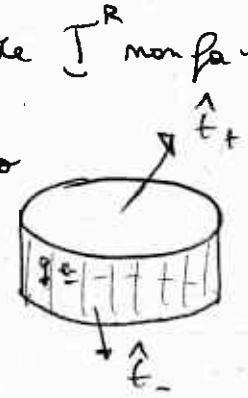
$$\underline{v} = (\tilde{v}_\alpha - x_3 \eta_{,\alpha}) \underline{e}_\alpha + \eta \underline{e}_3$$

PLV:

$$\int_P \underline{T}^{A} \cdot \nabla \underline{v} = \left[\text{per ipotesi } \underline{T}^R \cdot \underline{\xi} = 0 \text{, d'altronde } \underline{T}^R \text{ non fa lavoro} \right]$$

$$= \int_P \underline{b} \cdot \underline{v} + \int_{\partial P} \underline{t} \cdot \underline{v}$$

Solo sul bordo si hanno σ vincoli σ carichi assegnati



$$\int_P T^A_{\alpha\beta} \left(\tilde{v}_{\alpha,\beta} - x_3 \eta_{,\alpha\beta} \right) =$$

$$= \int_S \left(\int_{-h}^{h} T_{\alpha\beta} \, dx_3 \right) \tilde{v}_{\alpha,\beta} - \int_S \left(\int_{-h}^{h} T_{\alpha\beta} \, x_3 \, dx_3 \right) \eta_{,\alpha\beta} =$$

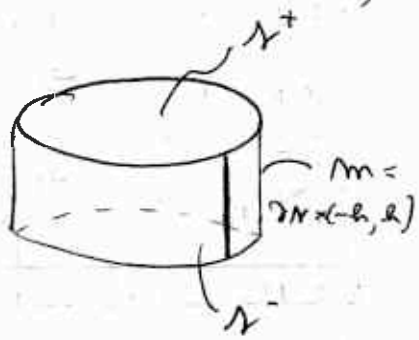
$$= \int_{\mathcal{N}} \left(N_{\alpha\beta}\, \tilde{\sigma}_{\alpha,\beta} - M_{\alpha\beta}\, \eta_{,\alpha\beta} \right) \quad \text{[int. per parti]} =$$

$$= \int_{\mathcal{N}} \Big[\left(N_{\alpha\beta} \tilde{\sigma}_\alpha \right)_{,\beta} \underbrace{- N_{\alpha\beta,\beta}\, \tilde{\sigma}_\alpha} - \left(M_{\alpha\beta}\, \eta_{,\alpha} \right)_{,\beta} + \underbrace{M_{\alpha\beta,\beta}\, \eta_{,\alpha}} \Big]$$

[eqn. th. equilib.

$$= \int_{\mathcal{N}} \left(-N_{\alpha\beta,\beta}\, \tilde{\sigma}_\alpha + M_{\alpha\beta,\beta}\, \eta_{,\alpha} \right) + \int_{\partial\mathcal{N}} \left(N_{\alpha\beta}\, n_\beta\, \tilde{\sigma}_\alpha - M_{\alpha\beta}\, n_\beta\, \eta_{,\alpha} \right) =$$

[2° membro]



$$\int_{\mathcal{N}} \int_{-h}^{h} \underline{b} \cdot \left[\left(\tilde{\sigma}_\alpha - x_3\, \eta_{,\alpha} \right) \underline{e}_\alpha + \eta\, \underline{e}_3 \right] +$$

$$\int_{\mathcal{N}^+ \cup \mathcal{N}^-} \hat{\underline{t}} \cdot \left[\left(\tilde{\sigma}_\alpha - x_3\, \eta_{,\alpha} \right) \underline{e}_\alpha + \eta\, \underline{e}_3 \right] + \int_{\partial\mathcal{N}} \int_{-h}^{h} \hat{\underline{t}} \cdot \left[\left(\tilde{\sigma}_\alpha - x_3\, \eta_{,\alpha} \right) \underline{e}_\alpha + \eta\, \underline{e}_3 \right]$$

Tutto porta a:

$$= \int_{\mathcal{N}} \Big[\tilde{\sigma}_\alpha \underbrace{\left(\int_{-h}^{h} b_\alpha \, dx_3 + \hat{t}_\alpha^+ + \hat{t}_\alpha^- \right)}_{q_\alpha} + \eta \overbrace{\left(\int_{-h}^{h} b_3 \, dx_3 + \hat{t}_3^+ + \hat{t}_3^- \right)}^{q_3} \Big] +$$

$$- \int_{\mathcal{N}} \eta_{,\alpha} \underbrace{\left(\int_{-h}^{h} x_3\, b_\alpha \, dx_3 + h \left(\hat{t}_\alpha - \hat{t}_\alpha^- \right) \right)}_{m_\alpha} +$$

$$+ \int_{\partial\mathcal{N}} \left[\tilde{\sigma}_\alpha \int_{-h}^{h} \hat{t}_\alpha \, dx_3 + \eta \int_{-h}^{h} \hat{t}_3 \, dx_3 - \eta_{,\alpha} \int_{-h}^{h} x_3\, \hat{t}_\alpha \, dx_3 \right] =$$

$$= \int_{\mathcal{N}} \left(q_\alpha\, \tilde{\sigma}_\alpha + q_3\, \eta - \eta_{,\alpha}\, m_\alpha \right) + \int_{\partial\mathcal{N}} \left(\tilde{\sigma}_\alpha \hat{f}_\alpha + \eta\, \hat{f}_3 - \eta_{,\alpha} \hat{c}_\alpha \right)$$

28/04/09

$$\int_P \underline{T}^A \nabla \underline{v} = \int_P \underline{b} \cdot \underline{v} + \int_{\partial P} \hat{\underline{t}} \cdot \underline{v}$$

$$\int_N \left(-N_{\alpha\beta,\beta}\, \tilde{v}_\alpha + M_{\alpha\beta,\beta}\, \eta_{,\alpha}\right) + \int_{\partial N} \left(N_{\alpha\beta} n_\beta \tilde{v}_\alpha - M_{\alpha\beta} n_\beta \eta_{,\alpha}\right) =$$

$$= \int_N \left(q_\alpha \tilde{v}_\alpha + q_3 \eta - s_\alpha \eta_{,\alpha}\right) + \int_{\partial N} \left(\hat{f}_\alpha \tilde{v}_\alpha + \hat{f}_3 \eta + \right.$$

$$\left. - \hat{c}_\alpha \eta_{,\alpha}\right)$$

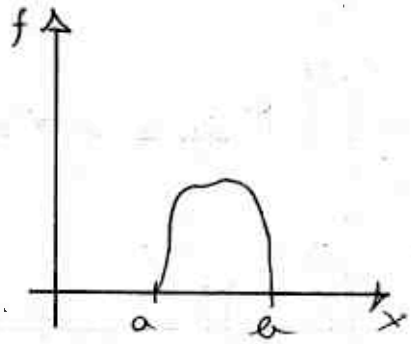
$N_{\alpha\beta}$ α $\tilde{u}_1, \tilde{u}_2$; $M_{\alpha\beta}$ w

La si può suddividere in 2 eq.

$$\int_N \left(-N_{\alpha\beta,\beta}\, \tilde{v}_\alpha\right) + \int_{\partial N} N_{\alpha\beta} n_\beta \tilde{v}_\alpha = \int_N q_\alpha \tilde{v}_\alpha + \int_{\partial N} \hat{f}_\alpha \tilde{v}_\alpha$$

$$-\int_N \left(N_{\alpha\beta,\beta} + q_\alpha\right) \tilde{v}_\alpha + \int_{\partial N} \left(N_{\alpha\beta} n_\beta - \hat{f}_\alpha\right) \tilde{v}_\alpha = 0$$

Se abbiamo una f. arbitraria non nulla in a - b, si costruisce fun. exp. regolare



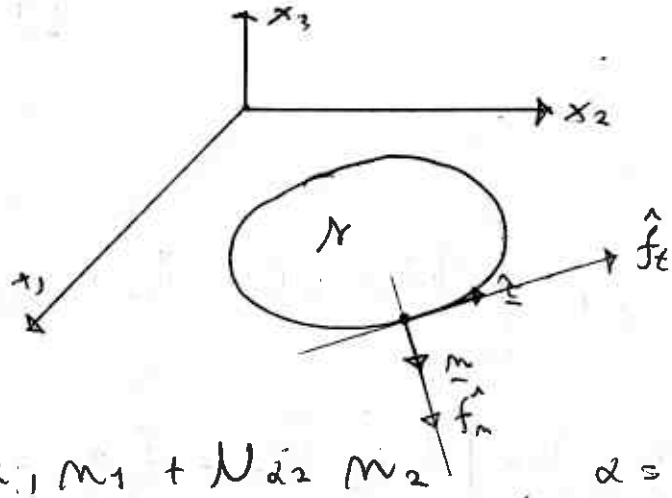
$$N_{\alpha\beta,\beta} + q_\alpha = 0 \quad \text{in } N, \quad \alpha = 1,2$$

Per esp.: $N_{11,1} + N_{12,1} + q_1 = 0$

$N_{21,1} + N_{22,2} + q_2 = 0$

Cond. essenziali: assegno $\tilde{u}_\alpha$, $\alpha = 1,2$

" naturali: $\hat{f}_\alpha$, $\alpha = 1,2$ quindi $N_{\alpha\beta} n_\beta = \hat{f}_\alpha$



$N_{\alpha\beta}\, n_\beta = \hat{f}_\alpha$ corrisp. a $\underline{T}\,\underline{n} = \hat{f}$
[2D] [3D]

Esprimiamo in termini di $\underline{t}, \underline{n}$
(+ senso, risp. alla curva)

$N_{\alpha\beta}\, n_\beta = (\underline{N}\,\underline{n})_\alpha$

$N_{\alpha 1} n_1 + N_{\alpha 2} n_2$, $\alpha = 1,2$. Assegnamo le forze

$N_{nn} = \hat{f}_n$; $N_{tn} = \hat{f}_t \rightarrow N_{tn} = \underline{N}\,\underline{n}\cdot\underline{t} = N_{\alpha\beta}\, n_\beta\, t_\alpha$

$\downarrow$

$\underline{N}\,\underline{n}\cdot\underline{n} = N_{\alpha\beta}\, n_\alpha\, n_\beta$

Possiamo in alternativa assegnare spostamenti:
$\tilde{u}_n$ e $\tilde{u}_t$

Si assegna $\tilde{u}_n = \hat{u}_n$, $\tilde{u}_t = \hat{u}_t$ (alternativa); $N_{nn} = \hat{f}_n$, $N_{tn} = \hat{f}_t$

Ordinando:

$$\int_{\mathcal{N}} M_{\alpha\beta,\beta}\, \eta_{,\alpha} + \int_{\partial\mathcal{N}} \left(-M_{\alpha\beta} n_\beta\, \eta_{,\alpha}\right) = \int_{\mathcal{N}} \left(q_3 \eta - \mathcal{A}_\alpha \eta_{,\alpha}\right) + \int_{\partial\mathcal{N}} \left(\hat{f}_3 - \hat{C}_\alpha \eta_{,\alpha}\right) =$$

$$= \int_{\mathcal{N}} \left(\left(M_{\alpha\beta,\beta} + \mathcal{A}_\alpha\right)\eta_{,\alpha} - q_3 \eta\right) + \int_{\partial\mathcal{N}} \left(\left(\hat{C}_\alpha - M_{\alpha\beta} n_\beta\right)\eta_{,\alpha} - \hat{f}_3\, \eta\right) = 0$$

$$\int_{\mathcal{N}} \left[\left(\left(M_{\alpha\beta,\beta} + \mathcal{A}_\alpha\right)\eta\right)_{,\alpha} - \left(M_{\alpha\beta,\alpha\beta} + \mathcal{A}_{\alpha,\alpha} + q_3\right)\eta\right] +$$

$$\left[\left(\hat{C}_\alpha - M_{\alpha\beta} n_\beta\right)\eta_{,\alpha} = \left(\hat{C}_\alpha - M_{\alpha\beta} n_\beta\right)\underline{e}_\alpha \cdot {}^s\nabla\eta\right.,$$

$$\eta = \eta(x_1, x_2)\,; \quad {}^s\nabla\eta = \eta_{,1}\, \underline{e}_1 + \eta_{,2}\, \underline{e}_2 =$$

$\downarrow$

GRADIENTE SUPERFICIALE di η

$\eta_{,s}\,\underline{\tau} + \eta_{,n}\,\underline{n}$; quindi $= \left(\hat{C}_\alpha - \Pi_{\alpha\beta}\, n_\beta\right) \underline{e}_\alpha \cdot \left(\eta_{,s}\,\underline{\tau} + \eta_{,n}\,\underline{n}\right) =$

$= \left(\hat{C}_t - \Pi_{tn}\right)\eta_{,s} + \left(\hat{C}_n - \Pi_{nn}\right)\eta_{,n}$ $\quad \left[\Pi_{\alpha\beta}\, n_\beta\, \underline{e}_\alpha = \underline{\Pi}\,\underline{n}\right]$]

$$+ \int_{\partial s} \left[\left(\hat{C}_t - \Pi_{tn}\right)\eta_{,s} + \left(\hat{C}_n - \Pi_{nn}\right)\eta_{,n} - f_3\,\eta \right] = 0$$

$$= \int_{\partial s} \left(\Pi_{\alpha\beta,\beta} + \mathcal{L}_\alpha\right) n_\alpha\, \eta - \int_s \left(\Pi_{\alpha\beta,\alpha\beta} + \mathcal{L}_{\alpha,\alpha} + q_3\right)\eta +$$

$$+ \int_{\partial s} \left(\left(\hat{C}_n - \Pi_{nn}\right)\eta_{,n} - f_3\,\eta \right) = 0$$

$$\left[\int_{\partial s} \left(\hat{C}_t - \Pi_{tn}\right)\eta_{,s} = \int_{\partial s} \left(\cancel{\left(\hat{C}_t - \Pi_{tn}\right)\eta}\,\Big)_{,s} - \left(\hat{C}_{t,s} - \Pi_{tn,s}\right)\eta \right)$$

↳ int. di una derivata, nullo se è curva regolare

$$\int_{\partial s} \left(\left(\hat{C}_n - \Pi_{nn}\right)\eta_{,n} - f_3\,\eta - \left(\hat{C}_{t,s} - \Pi_{tn,s}\right)\eta \right) = 0 \Bigg]$$

$$\left[\left(\Pi_{\alpha\beta,\beta} + \mathcal{L}_\alpha\right) n_\alpha = \underset{Q_1 n_1 + Q_2 n_2}{Q_\alpha\, n_\alpha} = Q_n \right].$$

$$- \int_s \left(\Pi_{\alpha\beta,\alpha\beta} + \mathcal{L}_{\alpha,\alpha} + q_3\right)\eta + \int_{\partial s} \Big[\left(Q_n + \Pi_{tn,s} \right) - \left(\hat{f}_3 + \hat{C}_{t,s} \right) \Big)\eta +$$

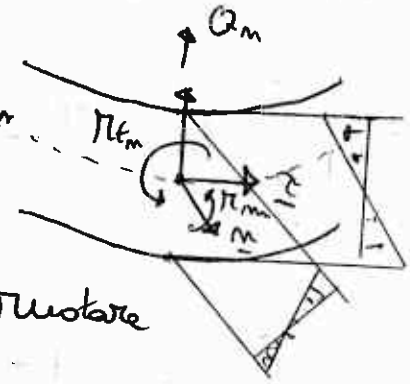
$$- \left(\Pi_{nn} - \hat{C}_n\right)\eta_{,n} \Big] = 0$$

$$\Pi_{\alpha\beta,\alpha\beta} + \mathcal{L}_{\alpha,\alpha} + q_3 = 0 \quad \text{in } s$$

- Assegno w oppure $Q_n + M_{tn,s} = \hat{f}_3 + \hat{C}_{t,s}$

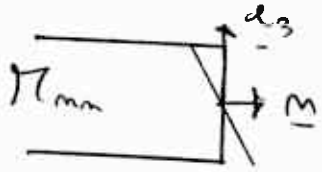
- Assegnamo $-w_n$ [Ret. sue bordo] oppure $M_{nn} = \hat{C}_n$

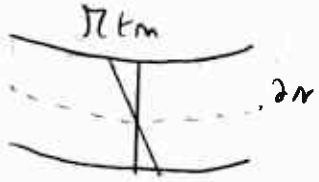
Ci aspettavamo solo f. assegnata. Invece abbiamo anche var. mom. torcente M_{tn} prodotto dalle tensioni τ.



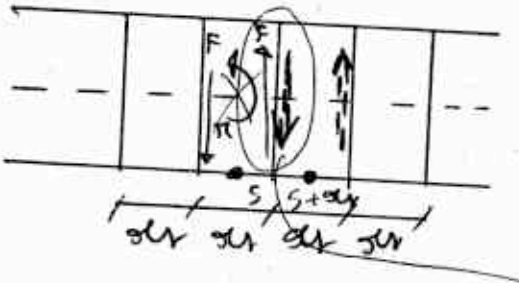
Il mom. flett. M_{nn} tende a far ruotare la fibra al bordo attorno a $\underline{\tau}$

Il mom. torcente M_{tn} tende a far ruotare M_{tn} intorno a $\underline{n}$.





Conv. bordo piastra.



$M_{tn} = F\,ds \rightarrow F = \frac{M_{tn}}{ds}$

↳ lo sost. con 2 forze a dist. di lavoro in

$M_{tn}(s+ds) = M_{tn}(s) + M_{tn,s}\,ds$

Quindi $F' = \frac{M_{tn}}{ds} + M_{tn,s}$ → sommando, si ha $M_{tn,s}$

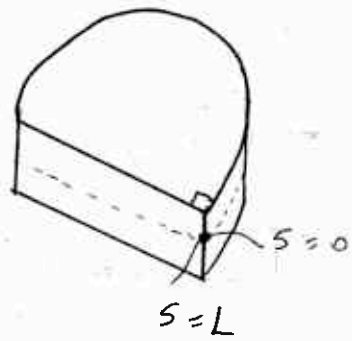
- $\hat{C}_{t,s}$ è distr. coppia torc. assegnata ⇝ raro, posiamo assegnare direttam. $\hat{f}_3$.

- α, α nella pratica non c'è mai.

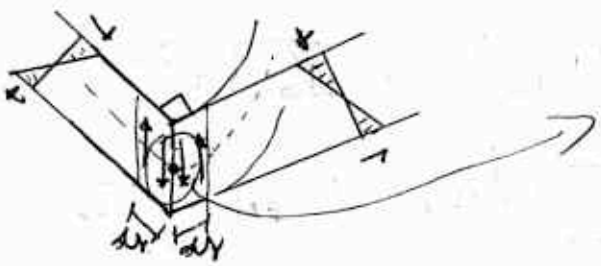
Se curva ∂s non è regolare)

Origine ha $s=0$ e $s=L$. Quindi

$$\int_{\partial s} \left((\hat{C}_t - M_{tn})\,\eta \right)_{,s} = \left[(\hat{C}_t - M_{tn})\,\eta \right]_0^L =$$



$$= \left(\hat{C}_t(L) - M_{tm}(L) \right) \eta(L) - \left(\hat{C}_t(0) - M_{tm}(0) \right) \eta(0)$$



Per simm. V_C e' M tric. ottorio, i termini le f. si sovrasso
Emergono FORZE CONCENTRATE

Forma delle eq.:

$$N_{11,1} + N_{12,2} + q_1 = 0$$. Per mat. trasv. isotropo:

$$2h \frac{E}{1-\nu^2} \left(\tilde{u}_{1,1} + \nu \tilde{u}_{2,2} \right)_{,1} + \frac{hE}{1+\nu} \left(\tilde{u}_{1,2} + \tilde{u}_{2,1} \right)_{,2} + q_1 = 0$$

$$N_{21,1} + N_{22,2} + q_2 = 0$$

$$\frac{hE}{1+\nu} \left(\tilde{u}_{1,2} + \tilde{u}_{2,1} \right)_{,1} + \frac{2hE}{1-\nu^2} \left(\tilde{u}_{2,2} + \nu \tilde{u}_{1,1} \right)_{,2} + q_2 = 0$$

$$M_{11,11} + M_{12,12} + M_{21,21} + M_{21,22} = M_{\alpha\beta,\alpha\beta}$$

$$M_{11} = -D\left(w_{,11} + \nu w_{,22} \right) ; \quad M_{22} = -D\left(w_{,22} + \nu w_{,11} \right) ;$$

$$M_{12} = -D(1-\nu) w_{,12}$$. Quindi

$$-D\left(w_{,11} + \nu w_{,22} \right)_{,11} - 2D(1-\nu) w_{,1212} - D\left(w_{,22} + \nu w_{,11} \right)_{,22} = q_3$$

$$= -D\left(w_{,1111} + 2w_{,1212} + w_{,2222} \right) \quad \text{[i termini con } \nu \text{ si annullano]}$$

$$\boxed{\Delta\Delta w = \frac{q}{D}}$$

Per mat. ortotropo:

$$\frac{2h\,E_{(1)}}{1-\nu_{(21)}\nu_{(12)}}\left(\tilde{u}_{1,1}+\nu_{(21)}\tilde{u}_{2,2}\right)_{,1}+2h\,G_{(12)}\left(\tilde{u}_{1,2}+\tilde{u}_{2,1}\right)_{,2}+q_1=0$$

$T_{12}=2G_{(12)}\,\varepsilon_{(12)}=G_{(12)}\left(\tilde{u}_{1,2}+\tilde{u}_{2,1}\right)\rightarrow$ integro tra $-h$ e h e ho $2h$

$$2h\,G_{(12)}\left(\tilde{u}_{1,2}+\tilde{u}_{2,1}\right)_{,1}+\frac{2h\,E_{(2)}}{1-\nu_{(12)}\nu_{(21)}}\left(\tilde{u}_{2,2}+\nu_{(12)}\tilde{u}_{1,1}\right)_{,2}+q_2=0$$

[se ortotr. tetrag. torna come prima]

Per il tr. isotr. $D=\frac{2}{3}h^3\frac{E}{1-\nu^2}$

Per il romerico, $D_{(1)}=\frac{2}{3}h^3\frac{E_{(1)}}{1-\nu_{(12)}\nu_{(21)}}$

[sost. $D_{(1)}$, $\nu_{(21)}$, $\nu_{(12)}$ nelle espr. di M_{11}, M_{22}]

$T_{12}=-2G_{(12)}\,\varepsilon_{12}=2G_{(12)}\,w_{,12}\,x_3$ (int. da $-h$ a h) (solo la parte che dà x_3)

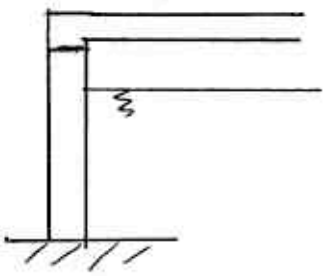
$$M_{12}=\int_{-h}^{h}x_3\,T_{12}=-\frac{2}{3}h^3\,2G_{(12)}\,w_{,12}=-D_{(12)}\,w_{,12}$$

$$-D_{(1)}\,w_{,1111}-\left(D_{(1)}\nu_{(21)}+D_{(2)}\nu_{(12)}+2D_{(12)}\right)w_{,1122}+$$
$$-D_{(2)}\,w_{,2222}+q=0$$

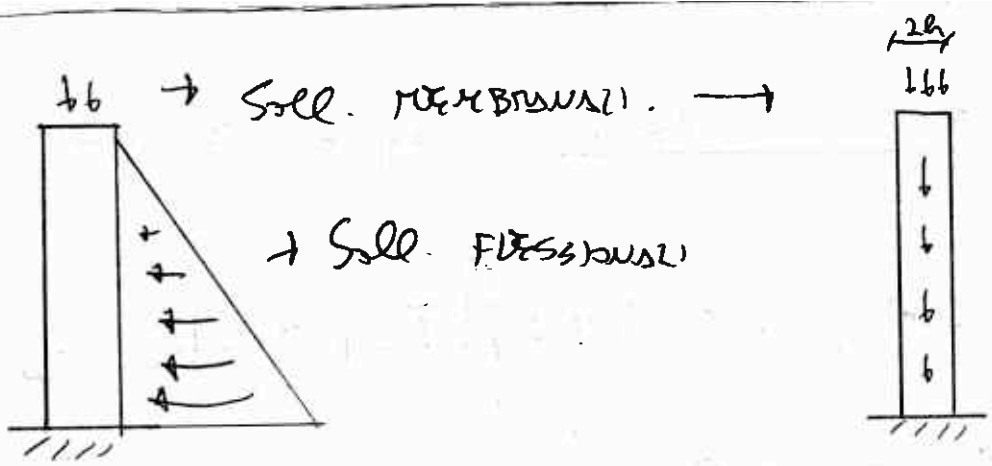
[eq. di equilibrio flex. molt. ort. rome.]

Formulazione variazionale del problema di equilibrio

Problema MEMBRANOSO



Ex: serbatoio con acqua, parete e solaio che poggia sopra di esso

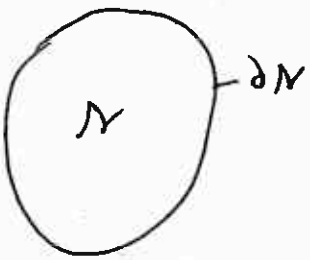


peso solaio
+
peso proprio parete

vincolo: spost. assegnato

$N_{\alpha\beta,\beta} + q_\alpha$ in Ω [posso assegnare F e r nello stesso punto se sono ⊥ (ex: ⟶ N)]

Cons. lavoro piastra divino:



$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ $\quad \tilde{u}_\alpha = \hat{u}_\alpha$ su $\partial\Omega_1$

$\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$ $\quad N_{\alpha\beta} n_\beta = \hat{f}_\alpha$ su $\partial\Omega_2$

Cons. $\tilde{\sigma}_\alpha = 0$ su $\partial\Omega_1$

$$\int_\Omega (N_{\alpha\beta,\beta} + q_\alpha)\tilde{\sigma}_\alpha = 0 \quad (\text{Per parti}) = \int_\Omega \left((N_{\alpha\beta}\tilde{\sigma}_\alpha)_{,\beta} + N_{\alpha\beta}\tilde{\sigma}_{\alpha,\beta} + q_\alpha \tilde{\sigma}_\alpha \right) = \int_{\partial\Omega_2} N_{\alpha\beta} n_\beta \tilde{\sigma}_\alpha \, ds +$$

$$- \int_\Omega \left(N_{\alpha\beta}\tilde{\sigma}_{\alpha,\beta} - q_\alpha \tilde{\sigma}_\alpha \right) =$$

$$\int_{\partial\Omega_2} \hat{f}_\alpha \hat{\sigma}_\alpha + \int_\Omega \left(q_\alpha \tilde{\sigma}_\alpha - N_{\alpha\beta}\tilde{\sigma}_{\alpha,\beta} \right) = 0$$

Det $\tilde{u}_1$, $\tilde{u}_2$ che risolv. eq. con cond. assegnate.

$$M_{\alpha\beta,\alpha\beta} + t_{\alpha,\alpha} + q_3 = 0 \quad \text{in } \Omega$$

$$\left.\begin{aligned} w &= \hat{w} \\ w_{,n} &= \hat{w}_{,n} \end{aligned}\right\} \text{ su } \partial\Omega_1 \qquad \left.\begin{aligned} Q_n + M_{tn,s} &= \hat{f}_3 + \hat{c}_{t,s} \\ M_{nn} &= \hat{c}_n \end{aligned}\right\} \text{ su } \partial\Omega_2$$

$$\left.\begin{array}{l}\eta = 0 \\ \eta_{,n} = 0\end{array}\right\} \text{ su } \partial\Omega, \qquad \text{Molt. per eq. di campo:}$$

$$0 = \int_\Omega \left(M_{\alpha\beta,\alpha\beta} + \mathcal{L}_{\alpha,\alpha} + q_3 \right)\eta = \int_\Omega \Big[\left((M_{\alpha\beta,\beta} + \mathcal{L}_\alpha)\eta \right)_{,\alpha} +$$

$$- M_{\alpha\beta,\beta}\,\eta_{,\alpha} - \mathcal{L}_\alpha \eta_{,\alpha} + q_3 \eta \Big] =$$

$$= \int_{\partial\Omega_2} (M_{\alpha\beta,\beta} + \mathcal{L}_\alpha)\eta\, n_\alpha - \int_\Omega (M_{\alpha\beta}\eta_{,\alpha})_{,\beta} + \int_\Omega \Big(M_{\alpha\beta}\eta_{,\alpha\beta} +$$

$$- \mathcal{L}_\alpha \eta_{,\alpha} + q_3 \eta \Big) =$$

$$= \int_{\partial\Omega_2} Q_n \eta - \int_{\partial\Omega} M_{\alpha\beta} n_\beta \eta_{,\alpha} + \int_\Omega \left(M_{\alpha\beta}\eta_{,\alpha\beta} - \mathcal{L}_\alpha \eta_{,\alpha} + q_3 \eta \right) =$$

$$\Big[(\underline{M}\,\underline{n})_\alpha \eta_{,\alpha} = (\underline{M}\,\underline{n})_\alpha \left(\eta_{,s}\tau_\alpha + \eta_{,n} n_\alpha \right) =$$

$$= M_{tn}\eta_{,s} + M_{nn}\eta_{,n} \Big]$$

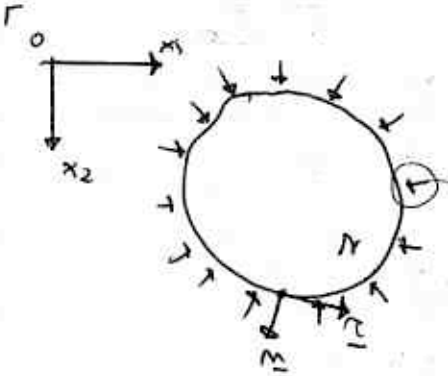
$$= \int_{\partial\Omega_2} Q_n \eta - \int_{\partial\Omega} \left(M_{tn}\eta_{,s} + M_{nn}\eta_{,n} \right) =$$

$$= \int_{\partial\Omega_2} \left(Q_n \eta - M_{nn}\eta_{,n} \right) + \int_{\partial\Omega_2} M_{tn,s}\,\eta$$

$$\Big[\int_{\partial\Omega} \left(M_{tn}\eta_{,s} \right) = \int_{\partial\Omega} \Big(\underbrace{\cancel{(M_{tn}\eta)_{,s}}}_{\to\ \partial\Omega \text{ regolare}} - M_{tn,s}\,\eta \Big) \Big] \quad \text{allora}$$

$$\int_{\partial\Omega_2} \left(\hat{f}_3 + \hat{c}_{t,s} \right)\eta - \hat{c}_n \eta_{,n} \Big) + \int_\Omega \left(M_{\alpha\beta}\eta_{,\alpha\beta} - \mathcal{L}_\alpha \eta_{,\alpha} + q_3 \eta \right) = 0$$

04/05/09



Poniamo forze rif. sempre al piano medio.

→ coll. TRASLAZIONE

$$N_{\alpha\beta,\beta} + q_\alpha = 0 \quad \alpha = 1,2$$

$$N_{11,1} + N_{12,2} + q_1 = 0$$

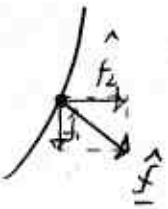
$$N_{21,1} + N_{21,2} + q_2 = 0$$

O assegno $\tilde{u}_1, \tilde{u}_2$ o in coord. locali $\tilde{u}_n, \tilde{u}_t$ [(. essenz.)]

// // forza per unità di lunghez. $\hat{f}_\alpha = N_{\alpha\beta} n_\beta$

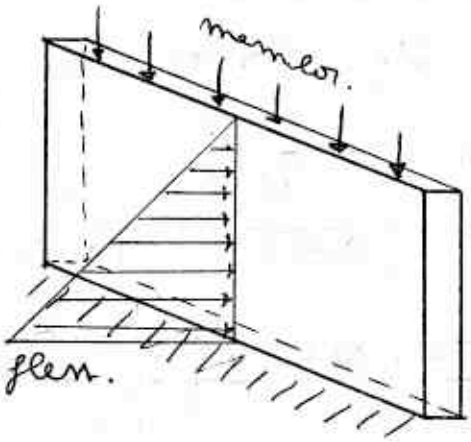
$\hat{f}_1 = N_{11} n_1 + N_{12} n_2$; $\hat{f}_2 = N_{21} n_1 + N_{22} n_2$

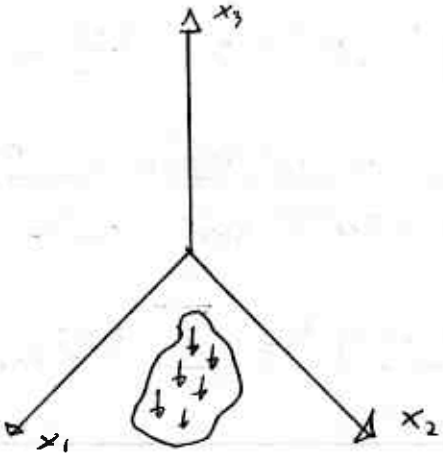
o meglio $\hat{f}_n = \underset{[N_n]}{N_{\alpha\beta} n_\alpha n_\beta}$; $\hat{f}_t = \underset{[N_{tn}]}{N_{\alpha\beta} n_\beta \tau_\alpha}$ [(. nat.]



Es:

Sollecitaz.:





Def. FLESSIONALI dovute a forze ⊥ a piano medio.

$$M_{\alpha\beta,\alpha\beta} + t_{\alpha,\alpha} + q_3 = 0$$

Assegno w, $Q_n + M_{tn,s} = \hat{f}_3 + \cancel{C_{t,n}}$ → non pres. in realtà oppure

// $-w_{,n}$, M_{nn}

Superficie si deforma, ha spost. in direzione 3.

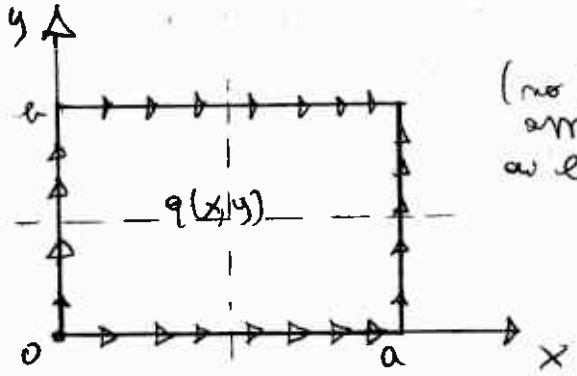
$$M_{\alpha\beta,\alpha\beta} + \cancel{t_{\alpha,\alpha}} + q_3 = 0$$

non pres. in realtà

↳ f. di volume dir. 3

(→)

Caso piastra rett. appoggiata sui 4 lati. Si ha:



(no M applicati ai bordi)

$y=0$; $\begin{cases} w(x,0)=0 \\ M_{yy}(x,0)=0 \end{cases}$

$y=b$; $\begin{cases} w(x,b)=0 \\ M_{yy}(x,b)=0 \end{cases}$

$x=0$; $\begin{cases} w(0,y)=0 \\ M_{xx}(0,y)=0 \end{cases}$ $\quad$ $x=a$; $\begin{cases} w(a,y)=0 \\ M_{xx}(a,y)=0 \end{cases}$ [cond. al contorno]

Se mat. ortotropo, $M_{yy} = -D_{(2)}(w_{,yy} + \nu_{(12)} w_{,xx})$

" " trasv. isotropo, $M_{yy} = -D(w_{,yy} + \nu w_{,xx})$

Le c. al contorno implicano $w_{,yy}(x,0)=0$;

$w_{,yy}(x,b)=0$; $w_{,xx}(0,y)=0$; $w_{,xx}(a,y)=0$

Cons. carico agente q trasversale (q_3).

Si scrive sol. nella forma di serie doppia:

$$\boxed{w(x,y) = \sum_{m,n}^{\infty} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}$$

[rispetta le c. al contorno]

$$w_{,xx} = -\frac{\pi^2}{a^2}\sum_{m,n}^{1\ldots\infty} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Quindi: [mat. ortotr.]

$$D_{(1)}\frac{\partial^4 w}{\partial x^4} + \left(2D_{(12)} + D_{(1)}\nu_{(21)} + D_{(2)}\nu_{(12)}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{(2)}\frac{\partial^4 w}{\partial y^4} = q$$

Esprimiamo prima $q(x,y)$ in serie doppia:

$$\boxed{q(x,y) = \sum_{m,n}^{1\ldots\infty} q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}$$

; molt. per $\sin\frac{p\pi x}{a} \sin\frac{q\pi x}{b}$ ambo i membri e integra $\int_0^a\int_0^b$:

Si annullano tutti tranne quando $n = p$ e $m = q$:

$$\frac{ab}{4} q_{pq} = \int_0^a \int_0^b q \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\boxed{q_{mn} = \frac{4}{ab}\int_0^a \int_0^b q \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}$$

Sostituiamo:

$$\sum_{m,n}^{1\dots\infty} \pi^4 \left(D_{(1)} \frac{m^4}{a^4} + \left(2D_{(12)} + \overbrace{D_{(1)}\nu_{(21)} + D_{(2)}\nu_{(12)}}^{\text{sono uguali}}\right)\frac{m^2 n^2}{a^2 b^2} + \right.$$

$$\left. + D_{(2)} \frac{n^4}{b^4} \right) w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = \sum_{m,n}^{1,\dots\infty} q_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) =$$

[Deve valere $\forall m, n$, quindi] $= q_{mn}$

Per trovare isotropo: $D_{(1)} = D = D_{(2)}$; $2D_{(12)} = 2D(1-\nu)$;
$D_{(1)}\nu_{(21)} + D_{(2)}\nu_{(12)} = 2D\nu$. Sost:

$$\pi^4 D \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 w_{mn} = q_{mn}$$

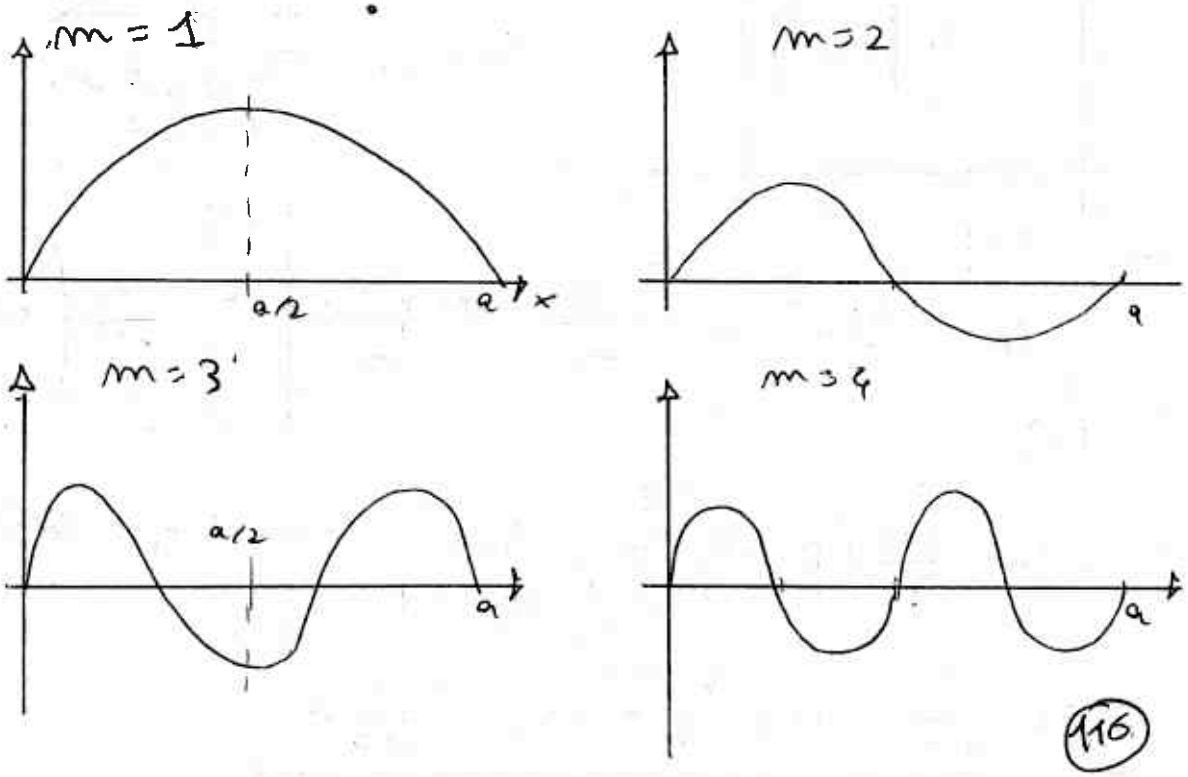
$$\boxed{w_{mn} = \frac{q_{mn}}{\pi^4 D\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2}}$$

La serie è determinata.

$\sin\frac{m\pi x}{a}$:

m pari: f asym risp. $a/2$

m dispari: f sym risp. $\frac{a}{2}$



Se carico è sym risp. a $x = \frac{a}{2}$, ci saranno solo gli m dispari
antisym " " " " " " " " " pari
" " " sym " " $y = \frac{a}{2}$ " " " " " dispari
" " " antisym " " " " " " " " " pari

Supponiamo carico uniforme

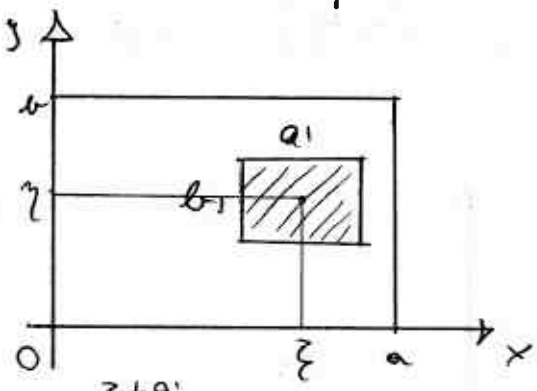
$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b \sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,q$$

$$\left[\int_0^a \sin\frac{m\pi x}{a} = \left[-\frac{a}{m\pi}\cos\frac{m\pi x}{a}\right]_0^a = -\frac{a}{m\pi}(-1-1) = \frac{2a}{m\pi}\right]$$

$$q_{mn} = \frac{4q}{\cancel{ab}}\,\frac{2\cancel{a}}{m\pi}\cdot\frac{2\cancel{b}}{n\pi} \Rightarrow$$ Per carico uniforme (con m, n dispari) $\boxed{q_{mn} = \frac{16\,q}{\pi^2 m n}}$

$$\boxed{w_{mn} = \frac{16\,q}{D\pi^6 m n\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}}$$

Carico uniforme su area rettangolare:



$$Q = q\,a_1 b_1$$

$$q_{mn} = \frac{4}{ab}\int_{\xi-\frac{a_1}{2}}^{\xi+\frac{a_1}{2}}\int_{\eta-\frac{b_1}{2}}^{\eta+\frac{b_1}{2}} \frac{Q}{a_1 b_1}\sin\frac{m\pi x}{a}\,\sin\frac{m\pi y}{b}$$

$$\left[\int_{\xi-\frac{a_1}{2}}^{\xi+\frac{a_1}{2}} \sin\frac{m\pi x}{a} = \left[-\frac{a}{m\pi}\cos\frac{m\pi x}{a}\right]_{\xi-\frac{a_1}{2}}^{\xi+\frac{a_1}{2}} = -\frac{a}{m\pi}\left[\cos\frac{m\pi}{a}\left(\xi+\frac{a_1}{2}\right) + \right.\right.$$

$$\left.\left. - \cos\frac{m\pi}{a}\left(\xi-\frac{a_1}{2}\right)\right]\right.$$; poi $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$ quindi

$$= \frac{a}{m\pi}\,2\sin\frac{m\pi\xi}{a}\,\sin\frac{m\pi a_1}{2a}\Bigg]$$

$$\boxed{q_{mn} = \frac{\cancel{4}^{\to 16}\, Q \leftarrow \cancel{4}\, \cancel{ab}}{\cancel{ab}\; a_1 b_1} \frac{}{\pi^2 m n} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \sin\frac{m\pi a_1}{2a} \sin\frac{n\pi b_1}{2b}}$$

Ci sono sia m pari che dispari [q non simmetrico]

•

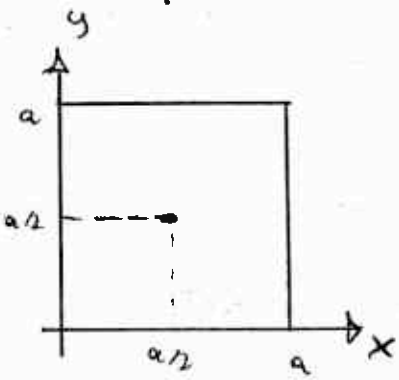
Si può fare tendere $a_1, b_1 \to 0$ con Q costante per avere carico concentrato in ξ, η.

$\lim_{x \to 0} \frac{1}{x} \sin \lambda x = \lambda$. Quindi sost.!

$$\boxed{q_{mn} = \frac{4Q}{ab} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}}$$

•

Cons. piastra quadrata app. sui 4 lati.



Supponiamo carico uniforme.

$$\boxed{w(x,y) = \frac{16\, q\, a^4}{\pi^6 D} \sum_{m,n} \frac{1}{mn(m^2+n^2)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}}$$

Spost. max al centro della piastra:

$$\boxed{w\left(\frac{a}{2}, \frac{a}{2}\right)} = \frac{16\, q\, a^4}{\pi^6 D} \sum_{m,n} \frac{\sin\frac{m\pi}{2} \sin\frac{n\pi}{2}}{mn(m^2+n^2)^2} = \left[\frac{16}{\pi^6} \simeq 1{,}664 \cdot 10^{-2}\right]$$

$$= 1{,}664 \cdot 10^{-2} \frac{q a^4}{D} \left(\underset{\left[\substack{m=1\\n=1}\right]}{\frac{1}{4}} - \underset{\left[\substack{m=1\\n=3}\right]}{\frac{1}{3 \cdot 5^2}} - \underset{\left[\substack{m=3\\n=1}\right]}{\frac{1}{3 \cdot 5^2}} + \underset{\left[\substack{m=3\\n=3}\right]}{\frac{1}{9 \cdot 18^2}} + \dots \right) =$$

$$= 1{,}664 \cdot 10^{-2} \frac{q a^4}{D} (0{,}25 - 0{,}00333 - 0{,}00333 + 0{,}00034 + \dots)$$

[largam. decrescenti, possiamo fermarci ai 1° 4] =

$$\boxed{= \frac{q a^4}{D}\, 0{,}00405}$$

Qui $M_{xx} = M_{yy} = -D\left(w_{,xx} + \nu w_{,yy}\right) = D\frac{16\, q a^{\cancel{4}\,2}}{\pi^{\cancel{6}\,4} \cancel{D}} \cdot$

$$\cdot \sum_{m,n}^{1\dots\infty} \left(\frac{1}{mn(m^2+n^2)^2} \left(\frac{m^2 \cancel{\pi^2}}{\cancel{a^2}} + \frac{n^2 \cancel{\pi^2}}{\cancel{a^2}} \nu \right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \right) =$$

$$= \frac{16\, q a^2}{\pi^4} \sum_{m,n} \frac{m^2 + \nu n^2}{mn(m^2+n^2)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}$$

In mezzeria si ha: $\left[\frac{16}{\pi^4} = 0{,}1643\right]$

$$M_{xx}\left(\frac{a}{2}, \frac{a}{2}\right) = 0{,}1643\, q a^2 \left(\underset{\left[\substack{m=1\\n=1}\right]}{\frac{1+\nu}{4}} - \underset{\left[\substack{m=1\\n=3}\right]}{\frac{1+9\nu}{3\cdot 10^2}} - \underset{\left[\substack{m=3\\n=1}\right]}{\frac{9+\nu}{3\cdot 10^2}} + \underset{\left[\substack{m=3\\n=3}\right]}{\frac{9(1+\nu)}{9\cdot 18^2}} + \dots \right)$$

$$= 0{,}1643\, q a^2 (1+\nu) \left(\frac{1}{4} - \frac{10}{300} + \frac{9}{2916} \dots \right) =$$

$$= 0{,}1643\, q a^2 (1+\nu) \left(0{,}25 - 0{,}03333 + 0{,}00309 + \dots \right)$$

[decrescono velocemente; anche qui primi 4 term] =

$$= \boxed{M_{xx}\left(\frac{a}{2}, \frac{a}{2}\right) = q a^2 (1+\nu)\, 0{,}03610}$$

M_{xy} per sym $= 0$

Carico concentrato al centro della piastra

$$q_{mn} = \frac{4Q}{a^2} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2}$$

$$w_{mn} = \frac{4Q}{\cancel{a^2}} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2} \frac{a^{\cancel{4}\,2}}{\pi^4 D(m^2+n^2)}$$

$$w(x,y) = \frac{4Q a^2}{\pi^4 D} \sum_{m,n} \frac{1}{(m^2+n^2)^2} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}$$

Massimo al centro:

$$w\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{4Q a^2}{\pi^4 D} \sum_{m,n} \frac{1}{(m^2+n^2)^2} \sin^2\frac{m\pi}{2} \sin^2\frac{n\pi}{2} =$$

$$= \frac{4Q a^2}{\pi^4 D} \left(\frac{1}{4} + \frac{1}{10^2} + \frac{1}{10^2} + \frac{1}{18^2} + \dots \right) =$$

$$= \frac{Qa^2}{D}\, 0{,}04106\left(0{,}25 + 0{,}01 + 0{,}01 + 0{,}00309 + \dots\right)$$ [stesso dim.]

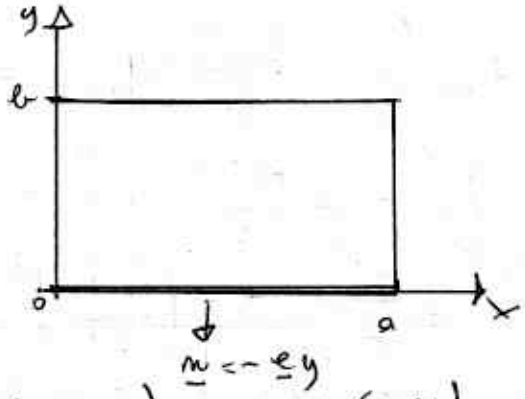
$$M_{xx}\left(\frac{a}{2};\frac{a}{2}\right) = \frac{4Q}{\pi^2}\sum_{m,n}\frac{m^2+n^2\nu}{(m^2+n^2)^2}\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2} =$$

$$= Q\cdot 0{,}40528\,(1+\nu)\left(0{,}25+0{,}10+0{,}02778+\dots\right)$$ [con 1° 4 term.]:

$$\simeq Q(1+\nu)\cdot 0{,}3778$$

Azioni piastra sul bordo.

Taglio ai bord. $V_n = Q_n + M_{tn,s}$

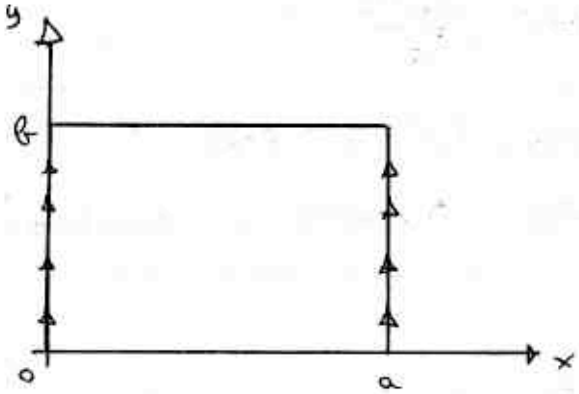


Ex: soll. su lato 0-a inferiore.

$$\boxed{V_y} = M_{yy,y} + M_{yx,x} + M_{xy,x} =$$

$$\left[Q_\alpha = M_{\alpha\beta,\beta} + \cancel{q_\alpha}\right] = -D\left(w_{,yy} + \nu w_{,xx}\right)_{,y} - 2D(1-\nu)\cdot w_{,xyx}$$

Su quel lato $w(x,0)=0$. Rimane $\boxed{-D\,w_{,yyy}}$



Piastra con 2 lati opposti appoggiati e scarichi (al cont.): 05/5/09

$w(0,y)=0;\ w_{,xx}(0,y)=0$

$w(a,y)=0;\ w_{,xx}(a,y)=0$

Supponiamo carico trasv. $q(x,y)$. Sol. è

$$w(x,y) = \underbrace{w_0(x,y)}_{\text{sol. prob. omogeneo}} + \underbrace{w_1(x,y)}_{\text{int. part. eq. non omogenea}}$$

$\Delta\Delta w_0 = 0;\quad \Delta\Delta w_1 = \frac{q}{D}$. Scegliamo:

$$w_0 = \sum_{m=1}^{\infty} f_m(y)\sin\frac{m\pi x}{a};\quad w_1 = \sum_{m=1}^{\infty} g_m(y)\sin\frac{m\pi x}{a}$$

che risp. le c. al cont.:

$$\frac{\partial^4 w_0}{\partial x^4} + 2\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^4 w_0}{\partial y^4} = \textcircled{1} \qquad \text{Sost. serie:}$$

$$\sum_{m=1}^{\infty}\left(\frac{m^4\pi^4}{a^4} f_m - 2\frac{m^2\pi^2}{a^2} f_m'' + f_m^{IV}\right)\sin\frac{m\pi x}{a} = 0$$

Deve essere nullo ogni termine; si ha

$$f_m^{IV} - 2\frac{m^2\pi^2}{a^2} f_m'' + \frac{m^4\pi^4}{a^4} f_m = 0 \text{ ; l'eq. caratt. è:}$$

$$\lambda^4 - 2\frac{m^2\pi^2}{a^2}\lambda^2 + \frac{m^4\pi^4}{a^4} = 0 \text{ . Risolv. risp. a } \lambda^2$$

$$\lambda^2 = \frac{m^2\pi^2}{a^2} \pm \sqrt{\left(\frac{m^2\pi^2}{a^2}\right)^2 - \frac{m^4\pi^4}{a^4}} = \frac{m^2\pi^2}{a^2} \text{ ; } \lambda = \pm\frac{m\pi}{a}$$

$$f_m(y) = (\bar{C}_1^m + \bar{C}_2^m y)e^{\frac{m\pi}{a}y} + (\bar{C}_3^m + \bar{C}_4^m y)e^{-\frac{m\pi}{a}y}$$

↳ con radici doppie le cost. sono messe così

+ anche sovvolte come:

$$f_m(y) = a_m \sinh\frac{m\pi y}{a} + b_m \cosh\frac{m\pi y}{a} +$$

$$+ c_m y \sinh\frac{m\pi y}{a} + d_m y \cosh\frac{m\pi y}{a}$$

Imponendo le 4 cond. al contorno sui 2 lati "non appoggiati" (vincolati qualsiasi modo) si ottengono le costanti.

Sol particolare: $\textcircled{1} = \frac{q}{D}$. Poniamo

$$q(x,y) = \sum_{m=1}^{\infty} q_m(y)\sin\frac{m\pi x}{a} \text{ ; premolt. e molt.}$$

per $\sin\frac{m\pi x}{a}$ e $\int_0^a$; rimane il termine

con $m = n$.

$$(121) \quad q_m(y) = \frac{2}{a}\int_0^a q(x,y)\sin\frac{m\pi x}{a}\,dx$$

$$\sum_{m=1}^{\infty}\left(g_m^{IV} - \frac{2m^2\pi^2}{a^2}\, g_m'' + \frac{m^4\pi^4}{a^4}\, g_m\right)\sin\frac{m\pi x}{a} =$$

$$= \sum_{m=1}^{\infty} \frac{q_m}{D}\sin\frac{m\pi x}{a}$$. Valida $\forall$ termine:

$$g_m^{IV} - 2\,\frac{m^2\pi^2}{a^2}\, g_m'' + \frac{m^4\pi^4}{a^4}\, g_m = \frac{q_m}{D} \leftarrow$$ g_m integr. particolare

$$q_m = \frac{2q}{a}\int_0^a \sin\frac{m\pi x}{a} = \frac{2q}{a}\left[-\frac{a}{m\pi}\cos\frac{m\pi x}{a}\right]_0^a =$$

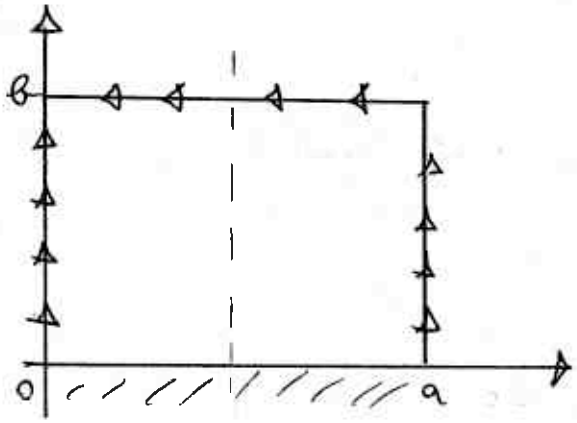
$\sin\frac{m\pi x}{a}$

Se carico è sym risp. a $x = \frac{a}{2}$ si cons. solo m. dispari.

$$= \frac{4q}{\cancel{\alpha}}\,\frac{\cancel{\alpha}}{m\pi}$$. Quindi per carico unif. $q_m = \frac{4q}{m\pi}$

$$g_m = \frac{4q}{m\pi}\,\frac{a^2}{m^4\pi^4} = \frac{4qa^4}{Dm^5\pi^5}$$

Quindi $$w(x,y) = \sum_{m=1}^{\infty}\left(a_m \sinh\frac{m\pi y}{a} + b_m\cosh\frac{m\pi y}{a} + c_m y\sinh\frac{m\pi y}{a} + d_m y\cosh\frac{m\pi y}{a} + g_m\right)\sin\frac{m\pi x}{a}$$



C. al contorno:

$w(x,0) = 0$; $w_{,y}(x,0) = 0$

$w(x,b) = 0$; $w_{,yy}(x,b) = 0$

$\downarrow$

$M_{yy}(x,b) = 0 =$

$= -D(w_{,yy} + \cancel{\nu w_{,xx}})$

Deriviamo la sol:

$$w_{,y} = \sum_{m=1}^{\infty} \left(\frac{m\pi}{a} a_m \cosh \frac{m\pi y}{a} + \frac{m\pi}{a} b_m \sinh \frac{m\pi y}{a} + \right.$$

$$+ c_m \left(\sinh \frac{m\pi y}{a} + y \frac{m\pi}{a} \cosh \frac{m\pi y}{a} \right) + d_m \left(\cosh \frac{m\pi y}{a} + \right.$$

$$\left. \left. + y \frac{m\pi}{a} \sinh \frac{m\pi y}{a} \right) \right) \sin \frac{m\pi x}{a}$$

$$w_{,yy} = \sum_{m=1}^{\infty} \left(\frac{m^2\pi^2}{a^2} a_m \sinh \frac{m\pi y}{a} + \frac{m^2\pi^2}{a^2} b_m \cosh \frac{m\pi y}{a} + \right.$$

$$+ c_m \left(2\frac{m\pi}{a} \cosh \frac{m\pi y}{a} + \frac{m^2\pi^2}{a^2} y \sinh \frac{m\pi y}{a} \right) +$$

$$\left. + d_m \left(2 \frac{m\pi}{a} \sinh \frac{m\pi y}{a} + \frac{m^2\pi^2}{a^2} y \cosh \frac{m\pi y}{a} \right) \right) \sin \frac{m\pi x}{a}$$

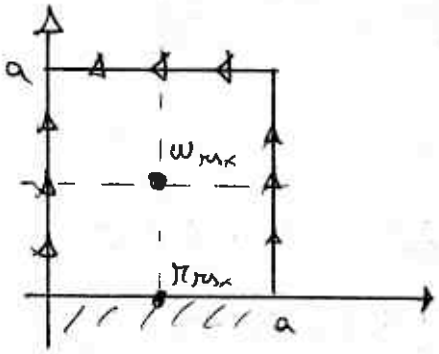
Applichiamo le c al contorno:

$$b_m + \frac{4qa^4}{\pi^5 m^5} = 0 \quad ; \quad \frac{m\pi}{a} a_m + d_m = 0$$

$$a_m \sinh \frac{m\pi b}{a} + b_m \cosh \frac{m\pi b}{a} + c_m b \sinh \frac{m\pi b}{a} +$$

$$+ d_m b \cosh \frac{m\pi b}{a} + \frac{4qa^4}{D\pi^5 m^5} = 0$$

Sist. 4 eq. in 4 inc. per det. le costanti



Piastra quad., vogliamo abbass. in mezzo:

$$w(x,y) = \sum_{m=1}^{\infty} \left(a_m \sinh \frac{m\pi y}{a} + \dots + \right.$$

$$\left. \frac{4a^4 q}{m^5 \pi^5 D} \right) \sin \frac{m\pi x}{a}$$

123 $m=1$: $w\left(\frac{a}{2}; \frac{a}{2}\right) = 0{,}002832 \frac{qa^4}{D}$

$m=3$: $w\left(\frac{a}{2};\frac{a}{2}\right) = 0{,}0004942 \ \frac{qa^4}{D}$

$m=5$: $w\left(\frac{a}{2};\frac{a}{2}\right) = 0{,}0000004161 \ \frac{qa^4}{D}$

Calcoliamo momenti in $\left(\frac{a}{2},0\right)$:

$$M_{yy}\left(\frac{a}{2},0\right) = -D\, w_{,yy}$$

$m=1$: $M_{yy}\left(\frac{a}{2},0\right) = -0{,}08788 \ qa^2$

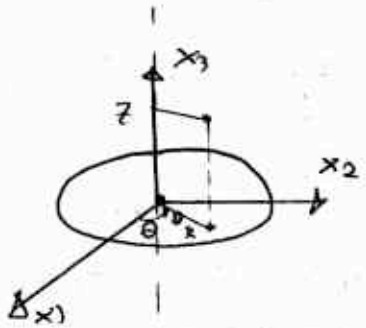
$m=3$: $M_{yy}\left(\frac{a}{2},0\right) = 0{,}004770 \ qa^2$ → Decresce al m e' meno marcato che con lo spett.

$m=5$: $M_{yy}\left(\frac{a}{2},0\right) = -0{,}001032 \ qa^2$

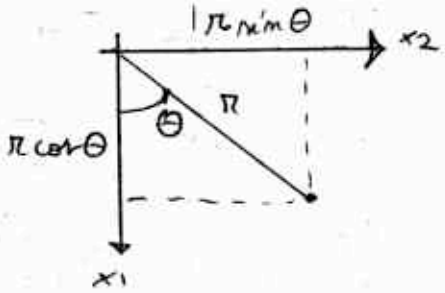
PIASTRE CIRCOLARI DI KIRCHHOFF-LOVE

Per piastra circolare con carico e sym assiale si hanno eq. stat.

Passiamo in coord. cilindriche

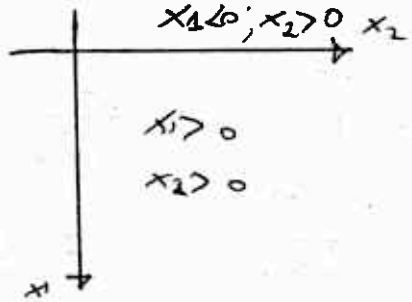


$$\begin{cases} x_1 = R\cos\theta \ ; \ x_2 = R\sin\theta \ ; \ x_3 = 0 \\ (r,\theta,z) = (z_1, z_2, z_3) \end{cases}$$



$$\begin{cases} r = \sqrt{x_1^2 + x_2^2} \ ; \ \theta \in [0, 2\pi[\\ \theta = \begin{cases} \operatorname{arctg}\frac{x_2}{x_1}, & \text{per } x_1>0, x_2 \geq 0 \\ \pi + \operatorname{arctg}\frac{x_2}{x_1}, & \text{per } x_1<0, x_2 \geq 0 \\ 2\pi + \operatorname{arctg}\frac{x_2}{x_1}, & \text{per } x_2<0 \end{cases} \\ z = x_3 \end{cases}$$

[trasf. di coordinate]

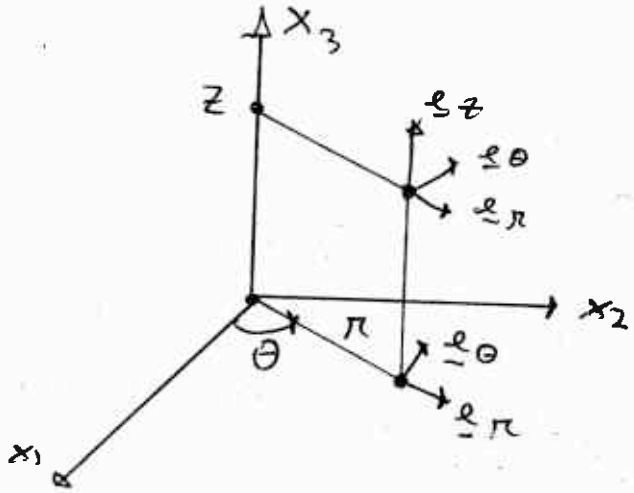


$$\frac{\partial r}{\partial x_1} = \frac{1}{2}\left(x_1^2 + x_2^2\right)^{-\frac{1}{2}} 2x_1 = \cos\theta$$

$$\frac{\partial r}{\partial x_2} = \frac{1}{2}\left(x_1^2 + x_2^2\right)^{-\frac{1}{2}} 2x_2 = \sin\theta$$

$$\frac{\partial\theta}{\partial x_1} = \frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\left(-\frac{x_2}{x_1^2}\right) = -\frac{x_2}{x_1^2+x_2^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial x_2} = \frac{\cos\theta}{r}$$



$$\begin{cases} \underline{e}_r = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2 \\ \underline{e}_\theta = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2 \\ \underline{e}_z = \underline{e}_3 \end{cases}$$

$$\underline{e}_{r,\theta} = \underline{e}_\theta \,;\; \underline{e}_{\theta,\theta} = -\underline{e}_r$$

$$\underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + u_3\underline{e}_3 = u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{e}_z$$

generico vett

Con vett. spost. se deform. è a sym assiale risp. a z non può esserci u_θ, allora $\underline{u} = u_r\underline{e}_r + u_z\underline{e}_z = \underline{u}(r, z)$, r α da θ.

In c. cart.: $\nabla\underline{u} = \underline{u}_{,i} \otimes \underline{e}_i = \frac{\partial\underline{u}}{\partial x_i} \otimes \underline{e}_i$

$\underline{u}(z_k(x_i))$ [u è f. delle c. cart. però è tramite delle c. cilindriche]

$$\nabla\underline{u} = \frac{\partial\underline{u}}{\partial z_k}\frac{\partial z_k}{\partial x_i} \otimes \underline{e}_i = \underline{u}_{,r} \otimes \left(\frac{\partial r}{\partial x_1}\underline{e}_1 + \frac{\partial r}{\partial x_2}\underline{e}_2 + \frac{\partial r}{\partial x_3}\underline{e}_3\right) + \underline{u}_{,\theta} \otimes \left(\frac{\partial\theta}{\partial x_1}\underline{e}_1 + \frac{\partial\theta}{\partial x_2}\underline{e}_2 + \frac{\partial\theta}{\partial x_3}\underline{e}_3\right) + \underline{u}_{,z} \otimes \left(\frac{\partial z}{\partial x_1}\underline{e}_1 + \frac{\partial z}{\partial x_2}\underline{e}_2 + \frac{\partial z}{\partial x_3}\underline{e}_3\right)$$

; le deriv. le abbiamo calcolate prima, quindi sostituiamo!

$$\nabla \underline{u} = \underline{u}_{,r} \otimes (\cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2) + \underline{u}_{,\theta} \otimes \left(-\frac{\sin\theta}{r}\, \underline{e}_1 + \frac{\cos\theta}{r}\, \underline{e}_2\right) + \underline{u}_{,z} \otimes \underline{e}_z \Rightarrow$$

$\underline{e}_r$ $\quad \frac{1}{r}\underline{e}_\theta$

$$= \underline{u}_r \otimes \underline{e}_r + \frac{1}{r}\, \underline{u}_{,\theta} \otimes \underline{e}_\theta + \underline{u}_{,z} \otimes \underline{e}_z$$

$$\nabla(\underbrace{u_r \underline{e}_r}_{} + \overset{u_\theta \underline{e}_\theta}{\vee} + u_z \underline{e}_z) = u_{r,r}\, \underline{e}_r \otimes \underline{e}_r + u_{z,r}\, \underline{e}_z \otimes \underline{e}_r +$$

$$+ \underline{u}_{\theta,r}\, \underline{e}_\theta \otimes \underline{e}_r = (u_{r,r}\, \underline{e}_r + \cancel{u_{\theta,r}}\, \underline{e}_\theta + u_{z,r}\, \underline{e}_z) \otimes \underline{e}_r +$$

$$+ \frac{1}{r}\Big(\cancel{u_{r,\theta}}\, \underline{e}_r + u_r\, \underline{e}_\theta + \cancel{u_{\theta,\theta}}\, \underline{e}_\theta - \cancel{u_\theta}\, \underline{e}_r +$$

$$+ \cancel{u_{z,\theta}}\, \underline{e}_z\Big) \otimes \underline{e}_\theta + (u_{r,z}\, \underline{e}_r + \cancel{u_{\theta,z}}\, \underline{e}_\theta + u_{z,z}\, \underline{e}_z) \otimes \underline{e}_z$$

Se c'è sym ambole scompaiono i termini con θ

Nella th di h, $\varepsilon_{zz} = 0$ ovvero $\nabla \underline{u} \cdot \underline{e}_z \otimes \underline{e}_z =$
$= u_{z,z} = 0$. Quindi $u_z = w(r)$.

Poi $\varepsilon_{zr} = 0$; $\varepsilon_{z\theta} = 0$

$$\varepsilon_{zr} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) \cdot \left(\underline{e}_r \otimes \underline{e}_z\right) = \nabla \underline{u} \cdot \frac{1}{2}\left(\underline{e}_r \otimes \underline{e}_z + \underline{e}_z \otimes \underline{e}_r\right)$$

$\varepsilon_{zr} = \frac{1}{2}\left(u_{r,z} + u_{z,r}\right) = 0$ quindi $u_{r,z} = -w_{,r}$

allora $u_r = u(r) - z\, w_{,r}$; $\varepsilon_{z\theta} = 0$ già soddisf.

α o θ $\Downarrow$

$$\boxed{\underline{u}(r,\theta,z) = u(r)\, \underline{e}_r + w(r)\, \underline{e}_z - z\, w_{,r}(r)\, \underline{e}_r}$$

5 $\mathrm{Div}\, \underline{N} = \underline{N}_{,\alpha}\, \underline{e}_\alpha = \underline{N}_{,1}\, \underline{e}_1 + \underline{N}_{,2}\, \underline{e}_2 =$

$$= \frac{\partial \underline{N}}{\partial z^\beta} \frac{\partial z^\beta}{\partial x^\alpha}\, \underline{e}_\alpha = \underline{N}_{,r}\, \underline{e}_r + \frac{1}{r}\, \underline{N}_{,\theta}\, \underline{e}_\theta$$

$$\varepsilon_{rr} = u_{,r} - z\, w_{,rr} \quad ; \quad \varepsilon_{\theta\theta} = \nabla \underline{u} \cdot \underline{e}_\theta \otimes \underline{e}_\theta$$

$$\left[\frac{1}{r} \underline{u}_{,\theta} \otimes \underline{e}_\theta = \frac{1}{r}\left(u_r \underline{e}_\theta - \cancel{u_\theta \underline{e}_r} \right) \otimes \underline{e}_\theta \right]$$

$$\varepsilon_{\theta\theta} = \frac{1}{r} u - z \frac{1}{r} w_{,r} \quad ; \quad \varepsilon_{r\theta} = 0 \quad ; \quad \varepsilon_{zz} = \varepsilon_{z\theta} = \varepsilon_{zr} = 0$$

$$T_{rr} = \frac{E}{1-\nu}\left(\varepsilon_{rr} + \nu \varepsilon_{\theta\theta} \right) = \frac{E}{1-\nu^2}\left(u_{,r} + \nu \frac{1}{r} u + \right.$$

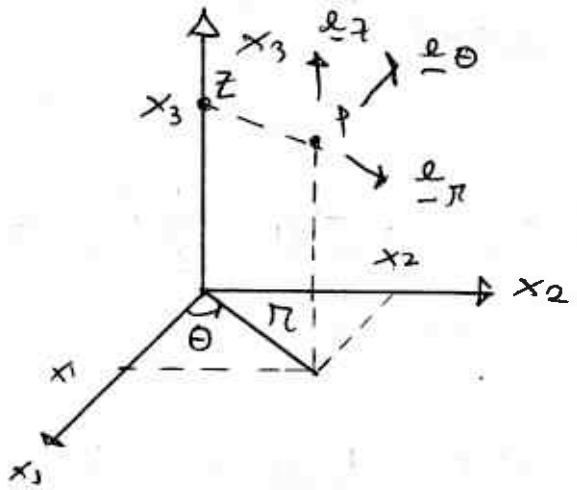
$$\left. - z \left(w_{,rr} + \frac{\nu}{r} w_{,r} \right) \right)$$

$$T_{\theta\theta} = \frac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu \varepsilon_{rr} \right) = \frac{E}{1-\nu^2}\left(\frac{1}{r} u + \nu u_{,r} + \right.$$

$$\left. - z \left(\frac{\nu}{r} w_{,r} + \nu w_{,rr} \right) \right)$$

$$T_{r\theta} = 0$$

11/05/09



$$x_1 = r \cos\theta$$
$$x_2 = r \sin\theta$$
$$x_3 = z$$

$$\underline{e}_r = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_1$$
$$\underline{e}_\theta = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2$$
$$\underline{e}_z = \underline{e}_3$$

$$r_{,1} = \cos\theta \quad ; \quad \theta_{,1} = -\frac{\sin\theta}{r} \quad ; \quad \underline{e}_{r,\theta} = \underline{e}_\theta$$
$$r_{,2} = \sin\theta \quad ; \quad \theta_{,2} = \frac{\cos\theta}{r} \quad ; \quad \underline{e}_{\theta,\theta} = \underline{e}_r$$

$$\nabla \underline{u} = \underline{u}_{,r} \otimes \underline{e}_r + \frac{1}{r} \underline{u}_{,\theta} \otimes \underline{e}_\theta + \underline{u}_{,z} \otimes \underline{e}_z$$

$$\underline{u} = \underline{u}(r, \theta, z) = u_r \underline{e}_r + u_\theta \underline{e}_\theta + u_z \underline{e}_z$$

$$\nabla \underline{u} = \left(u_{r,r} \underline{e}_r + u_{\theta,r} \underline{e}_\theta + u_{z,r} \underline{e}_z \right) \otimes \underline{e}_r +$$

$$+ \frac{1}{r}\left(u_{r,\theta} \underline{e}_r + u_r \underline{e}_\theta + u_{\theta,\theta} \underline{e}_\theta - u_\theta \underline{e}_r + \right.$$

$u_{z,\theta}\, \underline{e}_z) \otimes \underline{e}_\theta + (u_{r,z}\, \underline{e}_z + u_{\theta,z}\, \underline{e}_\theta + u_{z,z}\, \underline{e}_z) \otimes \underline{e}_z$

$\varepsilon_{ij} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) \cdot \underline{e}_i \otimes \underline{e}_j = \nabla u \cdot \frac{1}{2}(\underline{e}_i \otimes \underline{e}_j + \underline{e}_j \otimes \underline{e}_i)$

$\varepsilon_{rr} = \nabla \underline{u} \cdot \underline{e}_r \otimes \underline{e}_r = u_{r,r}$

$\varepsilon_{\theta\theta} = \frac{1}{r}(u_r + u_{\theta,\theta})$

$\varepsilon_{zz} = u_{z,z}$; $\varepsilon_{r\theta} = \nabla \underline{u} \cdot \frac{1}{2}(\underline{e}_r \otimes \underline{e}_\theta + \underline{e}_\theta \otimes \underline{e}_r) =$

$= \frac{1}{2}\left(u_{\theta,r} + \frac{1}{r} u_{r,\theta} - \frac{1}{r} u_\theta\right)$

$\varepsilon_{zr} = \nabla \underline{u} \cdot \frac{1}{2}(\underline{e}_z \otimes \underline{e}_r + \underline{e}_r \otimes \underline{e}_z) = \frac{1}{2}(u_{z,r} + u_{r,z})$

$\varepsilon_{z\theta} = \nabla \underline{u} \cdot \frac{1}{2}(\underline{e}_z \otimes \underline{e}_\theta + \underline{e}_\theta \otimes \underline{e}_z) = \frac{1}{2}\left(\frac{1}{r} u_{z,\theta} + u_{\theta,z}\right)$

Nella th. di Kirchhoff le limitazioni sono:

$\varepsilon_{zz} = 0$; $\varepsilon_{zr} = 0$; $\varepsilon_{z\theta} = 0$ e quindi

$u_{z,z} = 0$ $[u_z = w(r,\theta)]$; $u_{r,z} = -u_{z,r} = -w_{,r}$ integr.

e si ha $u_r = \tilde{u}_r(r,\theta) - z\, w_{,r}(r,\theta)$;

$u_{\theta,z} = -\frac{1}{r} u_{z,\theta} = -\frac{1}{r} w_{,\theta}$; $u_\theta = \tilde{u}_\theta(r,\theta) - z \frac{1}{r} w_{,\theta}(r,\theta)$

Inoltre si era assunta sym assiale $\Rightarrow$ comp. = in qualsiasi direz. $\Rightarrow$ reform. $\not\propto$ da θ.

$u_\theta = 0$, quindi si ha in caso di SIMMETRIA ASSIALE $\underline{u}(r,\theta,z) = (u(r) - z\, w_{,r}(r))\, \underline{e}_r \theta +$
$w(r)\, \underline{e}_z$.

Si ha:

$$\begin{cases} \varepsilon_{rr} = u_{r,r} = u_{,r} - z\, w_{,rr} \\ \varepsilon_{\theta\theta} = \frac{1}{r}(u - z\, w_{,r}) \\ \varepsilon_{r\theta} = 0 \end{cases}$$

Cons. mat. trasv. isotropo:

$$T_{rr} = \frac{E}{1-\nu^2}\left(E_{rr} + \nu E_{\theta\theta}\right) = \frac{E}{1-\nu^2}\left(u_{,r} + \frac{\nu}{r}u - z\left(w_{,rr} + \frac{\nu}{r}w_{,r}\right)\right)$$

$$T_{\theta\theta} = \frac{E}{1-\nu^2}\left(E_{\theta\theta} + \nu E_{rr}\right) = \frac{E}{1-\nu^2}\left(\frac{1}{r}u + \nu u_{,r} - z\left(\frac{1}{r}w_{,r} + \nu w_{,rr}\right)\right)$$

$$N_{rr} = \int_{-h}^{h} T_{rr}\,dz = \frac{2hE}{1-\nu^2}\left(u_{,r} + \frac{\nu}{r}u\right)$$

$$N_{\theta\theta} = \int_{-h}^{h} T_{\theta\theta}\,dz = \frac{2hE}{1-\nu^2}\left(\frac{1}{r}u + \nu u_{,r}\right)$$

$$N_{zr} = \int_{-h}^{h} T_{zr}\,dz$$ [non è stato da eq. cost.]

$$M_{rr} = \int_{-h}^{h} T_{rr}\,z\,dz = -D\left(w_{,rr} + \frac{\nu}{r}w_{,r}\right) \quad \text{con } D = \frac{2h^3}{3}\frac{E}{1-\nu^2}$$

$$M_{\theta\theta} = \int_{-h}^{h} T_{\theta\theta}\,z\,dz = -D\left(\frac{1}{r}w_{,r} + \nu w_{,rr}\right)$$

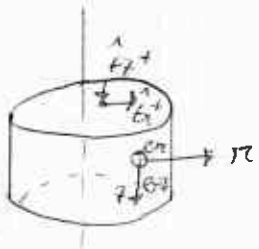
$$^{s}\mathrm{Div}\,\underline{N} = \underline{N}_{,r}\,\underline{e}_r + \frac{1}{r}\underline{N}_{,\theta}\,\underline{e}_\theta$$

In sym assiale $\underline{N} = N_{rr}\,\underline{e}_r \otimes \underline{e}_r + N_{\theta\theta}\,\underline{e}_\theta \otimes \underline{e}_\theta + N_{zr}\left(\underline{e}_z \otimes \underline{e}_r + \underline{e}_r \otimes \underline{e}_z\right)$. Quindi

$$^{s}\mathrm{Div}\,\underline{N} = N_{rr,r}\,\underline{e}_r + N_{zr,r}\,\underline{e}_z - \frac{1}{r}N_{\theta\theta}\,\underline{e}_r + \frac{1}{r}N_{rr}\,\underline{e}_r + \frac{1}{r}N_{zr}\,\underline{e}_z$$

Poiché $^{s}\mathrm{Div}\,\underline{N} + \underline{q} = 0$ allora

$$\underline{q} = \left(\int_{-h}^{h} b_r\,dz + \hat{t}_r^{+} + \hat{t}_r^{-}\right)\underline{e}_r + \left(\int_{-h}^{h} b_z\,dz + \hat{t}_z^{+} + \hat{t}_z^{-}\right)\underline{e}_z$$



1) $N_{rr,r} + \frac{1}{r}\left(N_{rr} - N_{\theta\theta}\right) + q_r = 0$

2) $N_{zr,r} + \frac{1}{r}N_{zr} + q_z = 0$

^{5} $\mathrm{Div}\,\underline{M} + \underline{\mathcal{L}} - \underline{N}\,\underline{e}_3 = \underline{0}$

3) $M_{rr,r} + \frac{1}{r}(M_{rr} - M_{\theta\theta}) + \mathcal{L}_r - N_{zr} = 0$ → anche pari a Q_r

con $\mathcal{L}_r = \int_{-h}^{h} b_r z\, dz + h(\hat{t}_r^+ - \hat{t}_r^-)$ (mom. delle f. che agiscono in dir. r, in realtà è 0)

La 2) e 3) collegate; N_{zr} è reattivo, ha elimin. x avere eq. in termini di spost. (quelli che compaiono in M_{rr} e $M_{\theta\theta}$)

Siamo interessati a sep. flex. (piano medio ecc).

$$-D\left(w_{,rrr} + \cancel{\frac{\nu}{r} w_{,rr}} - \cancel{\frac{\nu}{r^2} w_{,r}} + \frac{1}{r} w_{,rr} + \cancel{\frac{\nu}{r^2} w_{,r}} + \right.$$

$$\left. - \frac{1}{r^2} w_{,r} - \cancel{\frac{\nu}{r} w_{,rr}}\right) + \cancel{\mathcal{L}_r} = N_{zr}$$

$-D\left(w_{,rrr} + \left(\frac{1}{r} w_{,r}\right)_{,r}\right)$ oppure $-D\left(w_{,rr} + \frac{1}{r} w_{,r}\right)_{,r} = N_{zr}$

$-D\left(\frac{1}{r}(r w_{,r})_{,r}\right)_{,r} = N_{zr}$

La 2) si può scrivere $\frac{1}{r}(r N_{zr})_{,r} = -q$. Sostit.

$$\frac{1}{r}\left(r\left(\frac{1}{r}(r w_{,r})_{,r}\right)_{,r}\right)_{,r} = \frac{q}{D} \quad \text{[eq. sviluppata]}$$

Eq omogenea (ponendo $q = 0$) e dividendo

$$\left(r\left(\frac{1}{r}(r w_{,r})_{,r}\right)_{,r}\right)_{,r} = 0$$

quindi

$$\cancel{r}\left(\frac{1}{r}(r w_{,r})_{,r}\right)_{,r} = \frac{a_1}{r}$$

(div. per r) = (molt per r)

$$\frac{1}{\cancel{r}}(r w_{,r})_{,r} = r a_1 \ln r + a_2 r$$

Integr.

$$r w_{,r} = \left[\frac{r^2}{4}(2\ln r - 1)\right. \text{ scritto } = \frac{1}{4}\left(2r(2\ln r - 1)\right) + \quad 130$$

$$r^2\, 2\frac{1}{r}\Big) \rightsquigarrow \frac{1}{4}\left(4r\ln r - \cancel{2r} + \cancel{2r}\right)\Big] = \frac{a_1}{4} r^2 (2\ln r - 1)$$

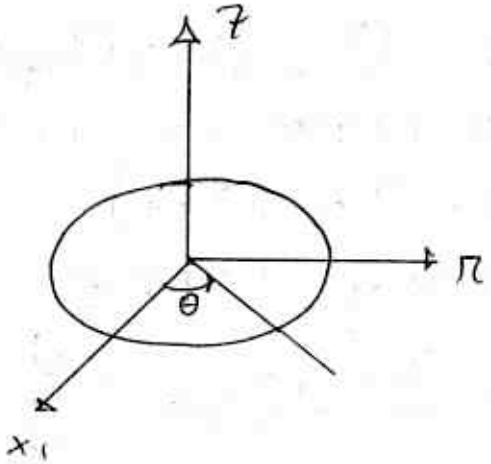
$$+ \frac{a_2}{2} r^2 + a_3$$

$$w_{,r} = \frac{a_1}{4} r (2\ln r - 1) + \frac{a_2}{2} r + \frac{a_3}{r}$$

$$w = \frac{a_1}{4} r^2 (\ln r - 1) + \frac{a_2}{4} r^2 + a_3 \ln r + a_4$$

$\downarrow$

Rappresenta lo spost., vogliamo valori FINITI.



Va chiesto che $a_3 = 0$

$$\lim_{r \to 0} \ln r = -\infty$$

[in a_1, r^2 è $\frac{1}{\infty^2}$ per $r \to 0$ che prevale sul log, no prob.]

$$w_{,rr} = \frac{a_1}{4}(2\ln r + 1) + \frac{a_2}{2}$$

Si ha:

$$M_{r,r} = -D\left(\frac{a_1}{4}(2\ln r + 1) + \frac{a_2}{2} + \nu\left(\frac{a_1}{4}(2\ln r - 1) + \frac{a_2}{2}\right)\right)$$

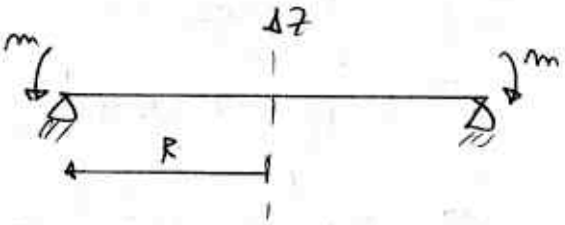
$\downarrow$

lo vogliamo finito al centro della piastra $\Rightarrow a_1 = 0$.

Quindi la sol. dell'omogenea è $\boxed{w(r) = C_1 r^2 + C_2}$

ESEMPI DI PIASTRE CIRCOLARI

Ex: <u>piastra circ. appoggiata sul contorno.</u>



Distribuz. di coppie sul bordo.

Cond. al contorno: $w(R) = 0$; $M_{nn}(R) = M_{rr}(R) = m$

$$C_1 R^2 + C_2 = 0$$

$$w_{,r} = 2C_1 r \; ; \; w_{,rr} = 2C_1$$

$$M_{rr} = -D\left(2C_1 + \nu 2C_1\right) = -2D(1+\nu)C_1$$

allora $-2D(1+\nu)C_1 = m$ e quindi

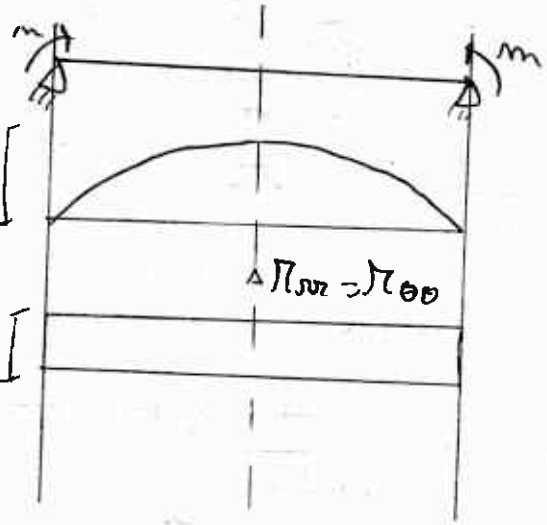
$$C_1 = \frac{m}{2D(1+\nu)} \quad ; \quad C_2 = \frac{m}{2D(1+\nu)} R^2$$ e allora

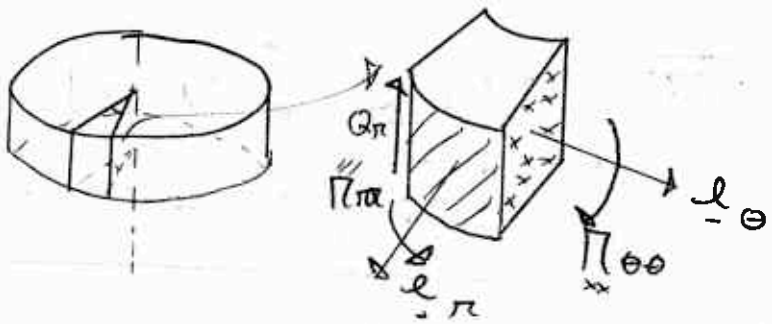
$$\boxed{w = \frac{m}{2D(1+\nu)}\left(R^2 - r^2\right)}$$

$$w_{,r} = \frac{m}{2D(1+\nu)}(-2r) \; ; \; w_{,rr} = -\frac{m}{D(1+\nu)}$$

$$M_{rr} = -D\left(w_{,rr} + \frac{\nu}{r} w_{,r}\right) = m$$

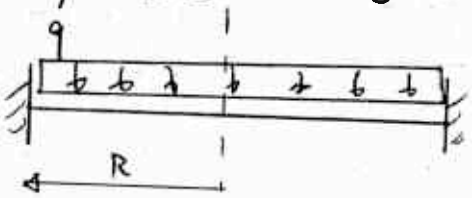
- Spost. ha and. parabolico e $w_{max} = \frac{mR^2}{2D(1-\nu)}$
- M_r è costante e pari a m
- $M_{\theta\theta} = M_{rr} = m$





Generalmente su faccia di $\perp$ r c'è anche taglio Q_r.

$$Q_r = M_{rr,r} + \frac{1}{r}\left(M_{rr} - M_{\theta\theta}\right) = 0$$ in questo caso



<u>Piastra circolare incastrata soggetta a carico unif.</u>

Ci vuole int. partic. da aggiungere a omog.

IP: $w^* = \frac{q r^4}{64 D} \xrightarrow{\text{DIR:}} w_{,r} = \frac{q r^3}{16 D}$; molt. per $r = \frac{q r^4}{64 D}$;

deriv: $\frac{q r^3}{4 D}$; divid. per r: $\frac{q r^2}{4 D}$; deriv. per r: $\frac{q r}{4 D}$;

molt. per r: $\frac{q r^2}{D}$; deriv e molt per $\frac{1}{r} = \frac{q}{D}$. C.V.D.

C. al contorno:

$$w(R) = C_1 R^2 + C_2 + q \frac{R^4}{64 D} = 0$$

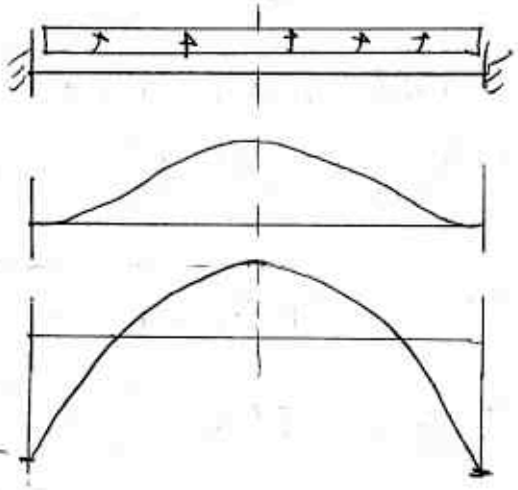
$$w_{,r}(R) = 2 C_1 R + \frac{q R^3}{16 D} = 0 \quad \text{(rotaz. nulla, normale } r\text{)}$$

Quindi $C_1 = -\frac{q R^2}{32 D}$; $C_2 = -\frac{q R^4}{64 D} + \frac{q R^4}{32 D} = \frac{q R^4}{64 D}$

Sol.: $\frac{q}{64 D}\left(r^4 + R^4 - 2 R^2 r^2\right)$

$$\boxed{w(r) = \frac{q}{64 D}\left(R^2 - r^2\right)^2}$$

(carico positivo)



$\frac{(1+\nu) q R^2}{16}$ ←

$$w_{max} = \frac{q R^4}{64 D}$$

$$w_{,r} = -\frac{q}{16 \cancel{64} D} \cancel{4}\left(R^2 - r^2\right) r$$

(poco + di - comp) ←

$$w_{,rr} = -\frac{q}{16 D}\left(R^2 - 3 r^2\right) \Rightarrow \boxed{M_{rr}} = -D\left(w_{,rr} + \frac{\nu}{r} w_{,r}\right) =$$

$$= \frac{q}{16}\left(R^2 - 3 r^2 + \nu\left(R^2 - r^2\right)\right) = \boxed{\frac{q}{16}\left(R^2(1+\nu) - r^2(3+\nu)\right)}$$

[per accidente $\nu \simeq \frac{1}{3}$].

$$\boxed{M_{\theta\theta}} = -D\left(w_{,r}\frac{1}{r} + \nu w_{,rr}\right) = \frac{q}{16}\left(\left(R^2 - r^2\right) + \nu\left(R^2 - 3 r^2\right)\right) = \boxed{\frac{q}{16}\left(R^2(1+\nu) - r^2(1+3\nu)\right)}$$

(andam. simile a M_{rr})

(133) $$\boxed{Q_r} = M_{rr,r} + \frac{1}{r}\left(M_{rr} - M_{\theta\theta}\right) = \boxed{\frac{q r}{2}}$$

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Se agisce carico q su piastra circolare,

$$w(r) = C_1 r^2 + C_2 + \frac{q r^4}{64 D}$$

$$w_{,r}(r) = 2C_1 r + \frac{q r^3}{16 D} \quad ; \quad w_{,rr}(r) = 2C_1 + \frac{3 q r^2}{16 D}$$

$$\frac{1}{r} w_{,r}(r) = 2C_1 + \frac{q r^2}{16 D}$$

$$M_{rr} = -D\left(w_{,rr} + \frac{\nu}{r} w_{,r}\right) = -D\left(2C_1(1+\nu) + \frac{q r^2}{16 D}(3+\nu)\right)$$

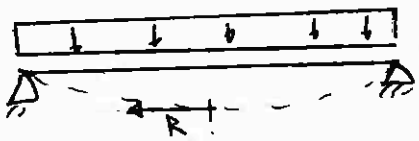
$$M_{\theta\theta} = -D\left(\frac{1}{r} w_{,r} + \nu w_{,rr}\right) = -D\left(2C_1(1+\nu) + \frac{q r^2}{16 D}(1+3\nu)\right)$$

$$Q_r = M_{rr,r} + \frac{1}{r}\left(M_{rr} - M_{\theta\theta}\right) = D\left[\frac{q r}{8 D}(3+\nu) - \frac{1}{r}\frac{q r^2}{16 D}(3+\nu + 1 - 3\nu)\right] = D\left[\frac{q r}{16 D}\left(6 + \cancel{2\nu} + 2 - \cancel{2\nu}\right)\right] = -\frac{q r}{2}$$

(taglio deve equilibrare il carico $\mathbf{F}$ & dalla cond. al contorno)

Taglio di Kirchhoff: $V_r = Q_r + M_{r\theta,\theta}$; qui $M_{r\theta} = 0$ però]

Ex:



$$w(R) = C_1 R^2 + C_2 + \frac{q R^4}{64 D} = 0$$

$$M_{rr}(R) = -D\left(2C_1(1+\nu) + \frac{q R^2}{16 D}(3+\nu)\right) = 0$$

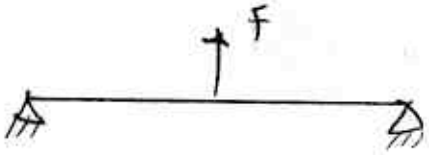
$$C_1 = -\frac{q R^2}{32 D}\frac{3+\nu}{1+\nu} \quad ; \quad C_2 = \frac{q R^4}{64 D}\frac{5+\nu}{1+\nu}$$

Sol:

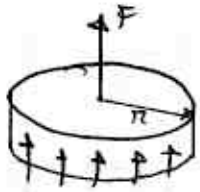
$$w(r) = \frac{-q}{64 D(1+\nu)}\left(2(3+\nu)(R^2 - r^2) - (1+\nu)(R^4 - r^4)\right)$$

$$\boxed{\begin{aligned} M_{rr}(r) &= -\frac{q}{16}(3+\nu)(R^2 - r^2) \\ M_{\theta\theta}(r) &= -\frac{q}{16}\left(2R^2(1-\nu) + (1+3\nu)(R^2 - r^2)\right) \end{aligned}}$$

Ex:



Piastra app. ai bordi con F conc. al centro. Isoliamo porzione piastra di raggio r,



I tagli devono equil. le forze:

$$Q_r\, 2\pi r + F = 0 \Rightarrow \boxed{Q_r = -\frac{F}{2\pi r}}$$

N_{zr} Sappiamo che $M_{rr,r} + \frac{1}{r}(M_{rr} - M_{\theta\theta}) = N_{zr}$ e

$-D\left(\frac{1}{r}(r w_{,r})_{,r}\right)_{,r} = N_{zr}$; ora sost. direttamente l'expr. di $N_{zr} = Q_r$ nota:

$$\left(\frac{1}{r}(r w_{,r})_{,r}\right)_{,r} = \frac{F}{2\pi D r}$$; integriamo:

$$\frac{1}{r}(r w_{,r})_{,r} = \frac{F}{2\pi D}\ln r + C_1$$; molt. per r:

$$(r w_{,r})_{,r} = \frac{F}{2\pi D} r \ln r + r C_1$$; integr.:

$$r w_{,r} = \frac{F}{2\pi D}\frac{1}{4} r^2 (2\ln r - 1) + C_1 \frac{r^2}{2} + C_2$$; dividi per r!

$$w_{,r} = \frac{F}{2\pi D}\frac{1}{4} r (2\ln r - 1) + C_1 \frac{r}{2} + \frac{C_2}{r}$$; Integr.:

$$w = \frac{F}{8\pi D} r^2 (\ln r - 1) + \frac{C_1}{4} r^2 + C_2 \ln r + C_3$$

[NOTA:

$$\int r \ln r\, dr = \int \ln r\, d\left(\frac{1}{2} r^2\right) = \frac{1}{2} r^2 \ln r - \int \frac{1}{2} r^2 \frac{1}{r}\, dr =$$

$$= \frac{1}{2} r^2 \ln r - \frac{1}{2}\frac{r^2}{2} + C$$]

Per avere spost. finito, $C_2 = 0$.

(135) $$w(r) = \frac{F}{8\pi D} r^2 (\ln r - 1) + \frac{C_1}{4} r^2 + C_3$$

Imponiamo le condizioni:

$w(R)=0$; $M_{rr}(R)=0$. Quindi:

$$\frac{C_1}{F} = \frac{F}{16\pi D(1+\nu)}\left(1-\nu+2(1+\nu)\ln R\right)$$

$$C_3 = -\frac{F}{16\pi D(1+\nu)} R^2 (3+\nu)$$. Sost. in w:

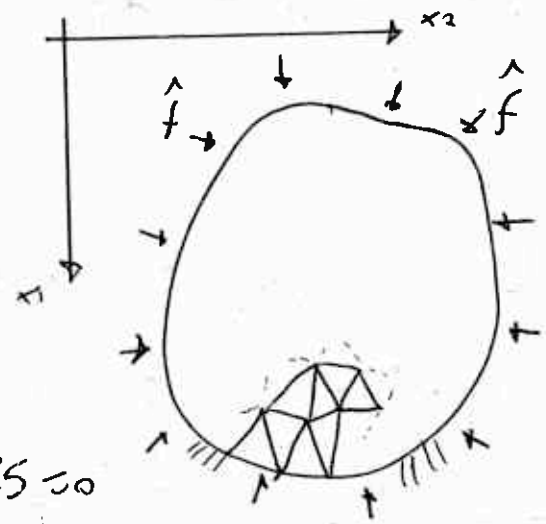
$$w = \frac{F}{16\pi D}\left(2(\ln R - \ln r)r^2 + \frac{3+\nu}{1+\nu}(r^2-R^2)\right)$$; quindi

$$\boxed{\begin{aligned} M_{rr} &= \frac{F}{4\pi}(1+\nu)(\ln r - \ln R) \\ M_{\theta\theta} &= \frac{F}{4\pi}\left((1+\nu)(\ln r - \ln R) - (1-\nu)\right) \\ Q_r &= M_{rr,r} + \frac{1}{r}(M_{rr} - M_{\theta\theta}) = \frac{F}{2\pi r} \end{aligned}}$$

H

STUDIO DEL PROBLEMA MEMBRANALE CON ELEMENTI FINITI

Cons. generica piastra (ci ref. al piano medio, come quando x la trave disegnamo solo l'asse).



Formul. variazionale:

$$\int_A \left(N_{\alpha\beta}\tilde{v}_{\alpha,\beta} - q_\alpha \tilde{v}_\alpha\right) dA - \int_{\partial A_2} \hat{f}_\alpha \tilde{v}_\alpha \, ds = 0$$

[in forma classica: $N_{\alpha\beta,\beta} + q_\alpha = 0$ in A

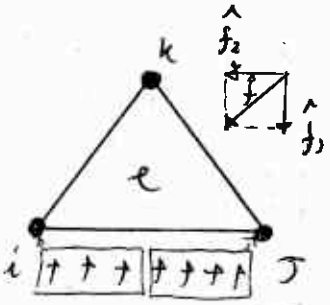
$u_\alpha = \hat{u}_\alpha$ su ∂A_1; $N_{\alpha\beta} n_\beta = f_\alpha$ su ∂A_2]

Vanno trovate $\tilde{u}_1$, $\tilde{u}_2$ che soddisf. c. essenz. ed eq. variaz. per tutte le scelte su ∂A_2. Arbitrarietà delle variaz. è alla base.

Suddividiamo corpo in elem. triangolari.
Conn. equil. singolo elemento e
contr. " corpo imponendo equil.

nodale, stesso principio del metodo degli spostam.
Conn. eq. elem. è:

$$\int_{\Omega^e} \left(N_{\alpha\beta}\, \tilde{u}_{\alpha,\beta} - q_\alpha \tilde{u}_\alpha \right) dA - \int_{\partial\Omega^e} f_\alpha\, \tilde{u}_\alpha\, dS = 0$$



Se e interno al corpo, al bordo le f sono le reaz. degli elem.
Se sul bordo o reaz. vinc., o f. assegnate, o liberi. Riduciamo tutte le forze a f. nodali. Esprimiamo $\tilde{u}_1^e$ e $\tilde{u}_2^e$ come prod. di coeff. per funz. di x_1, x_2.

$\tilde{u}_1^e = \sum \alpha_i \phi_i$; $N_{11} = \frac{2hE}{1-\nu^2}\left(\varepsilon_{11} + \nu \varepsilon_{22}\right)$

$\tilde{u}_2^e =$ "

Necessari polin. di grado 1

$f(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2$

$$\begin{cases} \tilde{u}_1^e(x_1, x_2) = S_{(i)1}\, \phi_i^e(x_1, x_2) + S_{(j)1}\, \phi_j^e(x_1, x_2) + S_{(k)1}\, \phi_k^e(x_1, x_2) \\ \left[\text{spostamenti del nodo } i\text{-esimo: } \underline{S}_{(i)} = \begin{bmatrix} S_{(i)1} \\ S_{(i)2} \end{bmatrix} \right] \\ \tilde{u}_2^e(x_1, x_2) = S_{(i)2}\, \varphi_i^e(x_1, x_2) + S_{(j)2}\, \varphi_j^e(x_1, x_2) + S_{(k)2}\, \varphi_k^e(x_1, x_2) \end{cases}$$

$\phi_i^e(x_{(i)1}, x_{(i)2}) = 1$; $\phi_j^e(x_{(i)1}, x_{(i)2}) = 0$

restrizione a parte del corpo di f. definita su tutto C

$\phi_k^e(x_{(i)1}, x_{(i)2}) = 0$;

$\phi_i^e(x_{(j)1}, x_{(j)2}) = 0$; $\phi_j^e(x_{(j)1}, x_{(j)2}) = 1$; $\phi_k^e(x_{(j)1}, x_{(j)2}) = 0$

$\phi_i^e(x_{k(1)}, x_{k(2)}) = 0$; $\phi_j^e(x_{k(1)}, x_{k(2)}) = 0$; $\phi_k^e(x_{k(1)}, x_{k(2)}) = 1$

3 cond per $\phi_i, \phi_j, \phi_k \Rightarrow$ 9 cond.

$$\phi_m^e(x_1, x_2) = a_{m0} + a_{m1} x_1 + a_{m2} x_2 \; ; \; m = i, j, k$$

Idem per le ψ. Le f. sono quindi note.

$$\underline{u}^e = \underline{\Phi}^{(e)} \underline{s}^e \quad \text{con} \quad \underline{u}^e = \begin{bmatrix} \tilde{u}_1^e \\ \tilde{u}_2^e \end{bmatrix}$$

$$\underline{\Phi}^e = \begin{bmatrix} \phi_i^e & 0 & \phi_j^e & 0 & \phi_k^e & 0 \\ 0 & \psi_i^e & 0 & \psi_j^e & 0 & \psi_k^e \end{bmatrix} ; \quad \underline{s}^e = \begin{bmatrix} \underline{s}_{(i)} \\ \underline{s}_{(j)} \\ \underline{s}_{(k)} \end{bmatrix}$$

$$\underline{v}^e = \underline{\Phi}^e \underline{\beta}^e$$

- Misura di deform. $\underline{\varepsilon}^e = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1 \partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}}_{\underline{D}} \underbrace{\begin{bmatrix} \tilde{u}_1^e \\ \tilde{u}_2^e \end{bmatrix}}_{\underline{u}^e} =$

$= \underline{D}\, \underline{\Phi}^e \underline{s}^e = \underline{B}^e \underline{s}^e$ → produce misura approx. della deform. ε

- Misura di sforzo: $\underline{\sigma}^e = \begin{bmatrix} N_{11} \\ N_{22} \\ 2N_{12} \end{bmatrix} = \underbrace{\frac{2hE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{bmatrix}}_{\text{matrice elastica } \underline{C}} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} =$

$= \underline{C}\, \underline{\varepsilon}^e = \underline{C}\, \underline{B}^e \underline{s}^e$

Quindi:

$$\int_{\Omega^e} \left(\underbrace{\underline{\sigma}^e}_{\underline{C}\underline{B}\underline{s}^e} \cdot \underline{B}^e \underline{\beta}^e - \underline{p}^e \cdot \underline{\Phi}^e \underline{\beta}^e \right) dA - \int_{\partial\Omega^e} \underline{f}^e \cdot \underline{\Phi}^e \underline{\beta}^e \, ds = 0$$

con $\underline{p}^e = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$; $\underline{f}^e = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. Si ha: (trasp. matr. nel pr. scalare)

$$\underbrace{\left(\int_{\Omega^e} \underline{B}^{eT} \underline{C}\, \underline{B}^e dA \right)}_{\underline{k}^e} \underline{s}^e \cdot \underline{\beta}^e - \underbrace{\left(\int_{\Omega^e} \underline{\Phi}^{eT} \underline{p}^e dA \right)}_{\underline{q}^e} \cdot \underline{\beta}^e +$$

$$-\underbrace{\left(\int_{\partial\Omega^c} \bar{\Phi}^{e^T} \underline{f}^e \, ds\right)}_{\underline{\pi}^e} \cdot \underline{\beta}^e = 0 \quad \text{allora}$$

$$\left(\underline{k}^e \underline{s}^e + \underline{q}^e - \underline{\pi}^e\right) \cdot \underline{\beta}^e = 0$$

Arbitrarietà è in $\underline{\beta}$ quindi $\boxed{\underline{k}^e \underline{s}^e + \underline{q}^e = \underline{\pi}^e}$

Come nel metodo degli sp. [eq. equil. elemento]

$\underline{k}^e \underline{s}^e$ sono le reaz. agli spost. nodali $\underline{s}^e$

$\underline{q}^e$ equilibrano carichi con spost. nodi nulli

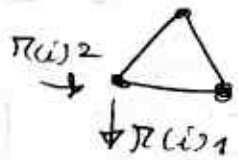
Suddividiamo $\underline{k}^e$ in sottom.: $\begin{bmatrix} \underline{k}_{ii}^e & \underline{k}_{ij}^e & \underline{k}_{ik}^e \\ \underline{k}_{ji}^e & \underline{k}_{jj}^e & \underline{k}_{jk}^e \\ \underline{k}_{ki}^e & \underline{k}_{kj}^e & \underline{k}_{kk}^e \end{bmatrix}$.

e molt. per $\underline{s}$ e sommo:

$$\begin{bmatrix} \underline{s}_i \\ \underline{s}_j \\ \underline{s}_k \end{bmatrix} + \begin{bmatrix} \underline{q}_i^e \\ \underline{q}_j^e \\ \underline{q}_k^e \end{bmatrix} = \begin{bmatrix} \underline{\pi}_i^e \\ \underline{\pi}_j^e \\ \underline{\pi}_k^e \end{bmatrix}$$

$$\underline{k}_{ii}^e \underline{s}_i + \underline{k}_{ij}^e \underline{s}_j + \underline{k}_{ik}^e \underline{s}_k + \underline{q}_i^e = \underline{\pi}_i^e$$

↳ ci fornisce $\pi_{(i)1}$ e $\pi_{(i)2}$ che contr. e mant. in equilibrio "e"



F. esterne = Σ reaz. che nodo su nodo esplica su tutti gli elem.

$$\underline{F}_{(i)} = \begin{bmatrix} F_{(i)1} \\ F_{(i)2} \end{bmatrix} = \sum_e \underline{\pi}_i^e$$

; unendo le eq. di equil. nodale di tutti i nodi,

si ha $\boxed{\underline{K}\,\underline{s} + \underline{q} = \underline{F}}$

[La diff. con il metodo degli spost. è che quello è esatto, questo è APPROX.]

Mancano le c. essenz. vincoli:

$\underline{S}_{(i)} = \hat{\underline{S}}_{(i)}$. Cancelliamo equaz. nodo i-esimo e sost. questa

Def. costanti, σ cost. nell'elemento.

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$$M_{\alpha\beta,\alpha\beta} + t_{\alpha,\alpha} + q_3 = 0 \quad , \text{ in } S$$

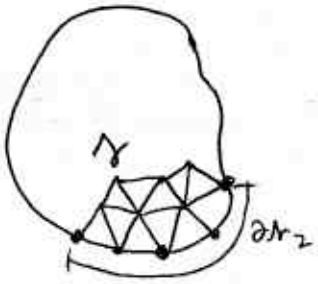
cui si associano le c. al contorno, suddiv. in 2 parti:

∂S_1: $w = \hat{w}$, $w_{,n} = \hat{w}_{,n}$

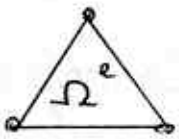
∂S_2: $Q_n + M_{tn,s} = \hat{f}_3 + \hat{C}_{T,s}$, $M_{nn} = \hat{C}_n$

A questa corrisponde: (forma variaz.)

$$\iint_S \left(M_{\alpha\beta}\, \eta_{,\alpha\beta} - t_\alpha \eta_{,\alpha} + q\eta \right) - \iint_{\partial S_2} \left(\left(\hat{f}_3 + \hat{C}_{t,s} \right) \eta - \hat{C}_n \eta_{,n} \right) = 0$$



(sul bordo triangoli approssimano contorno essendo multilineo)

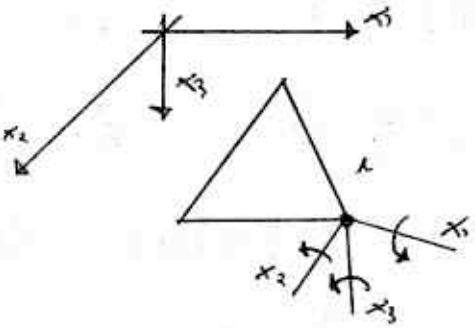


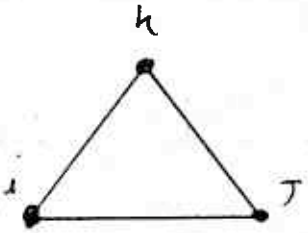
Form. variaz. con $S = \Omega^e$ e $\partial S = \partial\Omega^e$

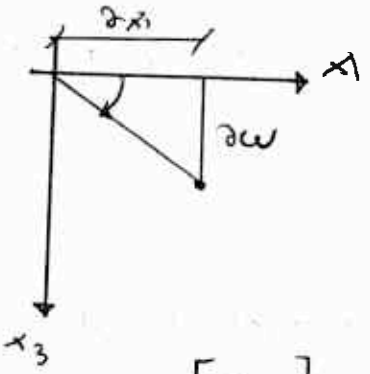
$\hat{f}_3$ e $\hat{C}_{t,s}$ note, essendo le reaz. sull'elem.

Incognita è spost.

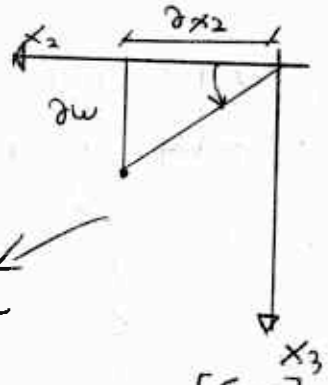
$$\underline{S}_{(i)} = \begin{bmatrix} S_{(i)1} \\ S_{(i)2} \\ S_{(i)3} \end{bmatrix} = \begin{bmatrix} w_{(i)} \\ \varphi_{(i)1} \\ \varphi_{(i)2} \end{bmatrix}$$







$$\varphi_2 = -\frac{\partial w}{\partial x_1} = -w_{,1}$$



$$\varphi_1 = \frac{\partial w}{\partial x_2} = w_{,2}$$

$$\underline{S}_e = \begin{bmatrix} \underline{S}_{(i)} \\ \underline{S}_{(j)} \\ \underline{S}_{(k)} \end{bmatrix} ; \quad [w^e] = \underline{\Phi}^e \underline{S}^e = [\phi_1^e, \phi_2^e, \dots \phi_9^e] \cdot \begin{bmatrix} S_{(i)1} \\ S_{(i)2} \\ \vdots \\ S_{(k)3} \end{bmatrix}$$

Sappiamo che ϕ sono polinomi. Qui essendo presenti derivate del 2° ordine e 9 cond. da applicare scegliamo polinomio CUBICO [10 coeff $[9\times3]$ in x_1, x_2, ne togliamo 1]:

$$\phi_m^e(x_1, x_2) = a_{0m} + a_{1m} x_1 + a_{2m} x_2 + a_{3m} x_1^2 + a_{4m} x_2^2 +$$
$$+ a_{5m} x_1^3 + a_{6m} x_1^2 x_2 + a_{7m} x_1 x_2^2 + a_{8m} x_2^3$$

[trascurato termine che molt. $x_1 x_2$]

$$w^e = w_{(i)} \phi_1^e + \varphi_{(i)1} \phi_2^e + \varphi_{(i)2} \phi_3^e + \dots + \varphi_{(k)2} \phi_9^e$$

(attento, sono fissi!)

Condizioni:

$$\phi_1^e(x_{(i)1}, x_{(i)2}) = 1 \qquad \phi_{1,\alpha}^e(x_{(i)1}, x_{(i)2}) = 0$$

$$\phi_1^e(x_{(j)1}, x_{(j)2}) = 0 \qquad \phi_{1,\alpha}^e(x_{(j)1}, x_{(j)2}) = 0 \quad \alpha = 1,2$$

$$\phi_1^e(x_{(k)1}, x_{(k)2}) = 0 \qquad \phi_{1,\alpha}^e(x_{(k)1}, x_{(k)2}) = 0$$

Analogo per ϕ_2; questa deve essere tale che:

(141) $\phi_{2,2}(x_{(i)1}, x_{(i)2}) = 1$

Quindi:

$$[w^e] = \underline{\Phi}^e \underline{s}^e$$. Introduco misura della deformazione!

$\underline{s}^e$ → gra mob

$$\underline{\varepsilon}^e = \begin{bmatrix} \chi_{11} \\ \chi_{22} \\ \chi_{12} \end{bmatrix} = \begin{bmatrix} \partial^2/\partial x_1^2 \\ \partial^2/\partial x_2^2 \\ \partial^2/\partial x_1 \partial x_2 \end{bmatrix} [w^e] = \underline{D}\, \underline{\Phi}^e \underline{s}^e = \underline{B}^e \underline{s}^e$$

Definiamo una misura di sforzo:

$$\underline{\sigma}^e = \begin{bmatrix} M_{11} \\ M_{22} \\ 2M_{12} \end{bmatrix} = -D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{bmatrix} \underline{\varepsilon}^e = \underline{C}\, \underline{B}^e \underline{s}^e$$

$\underline{C}$ → mat. elastica

$$M_{11} = -D(w_{,11} + \nu w_{,22}) \quad ; \quad M_{12} = -D(1-\nu)\, w_{,12}$$

$$[\eta^e] = \underline{\Phi}^e \underline{\beta}^e \quad ; \quad \eta_{,\alpha\beta} = \underline{B}^e \underline{\beta}^e$$

$$\int_{\Omega^e} \underline{C}\, \underline{B}^e \underline{s}^e \cdot \underline{B}^e \underline{\beta}^e - \int_{\Omega^e} \left([\mathcal{M}_\alpha] \cdot \underline{\Phi}_{,\alpha} \underline{\beta}^e - [q] \cdot \underline{\Phi}\, \underline{\beta}^e \right) +$$

Definiti per comodità $[\mathcal{M}_1]$, $[\mathcal{M}_2]$ e $[f_3 + C_{T,n}]$

$$- \int_{\partial\Omega^e} [f_3 + C_{T,n}] \cdot \underline{\Phi}^e \underline{\beta}^e - [C_n]\, \underline{\Phi}^e_{,n} \underline{\beta}^e \Big) = 0$$

$\left[\text{con } \underline{\Phi}_{,n} = \underline{\Phi}_{,1} n_1 + \underline{\Phi}_{,2} n_2 \right]$. Si può scrivere:

$$\underbrace{\left(\int_{\Omega^e} \underline{B}^{eT} \underline{C}\, \underline{B}^e \right)}_{\underline{k}^e} \underline{s}^e \cdot \beta^e - \int_{\Omega^e} \overbrace{\left(\underline{\Phi}^T_{,\alpha} [\mathcal{M}_\alpha] - \underline{\Phi}^{eT} [q] \right)}^{\underline{q}^e} \cdot \underline{\beta}^e +$$

$$-\int_{\partial\Omega^e} \underbrace{\left(\underline{\Phi}^{e^T} [f_3 + C_{t,s}] - \underline{\Phi}^{e^T}_{,n} \cdot [C_n] \right)}_{\underline{\pi}^e} \cdot \underline{\beta}^e = 0$$

Allora $\left(\underset{\text{OK}}{\underline{K}^e} \underset{\text{INC.}}{\underline{S}^e} + \underset{\text{OK}}{\underline{q}^e} - \underset{\text{INC.}}{\underline{\pi}^e} \right) \cdot \underset{\text{ARBITRARIO}}{\underline{\beta}^e} = 0$ quindi

equilibrio è dato da $\boxed{\underline{K}\,\underline{S}^e + \underline{q}^e = \underline{\pi}^e}$ (eq. a 9 comp.)

$$\underline{K}^e_{ii} \underline{S}_i + \underline{K}^e_{ij} \underline{S}_j + \underline{K}^e_{in} \underline{S}_n + \underline{q}^e_i = \underline{\pi}^e_i$$
$$\underline{K}^e_{ji} \underline{S}_i + \underline{K}^e_{jj} \underline{S}_j + \underline{K}^e_{jn} \underline{S}_n + \underline{q}^e_j = \underline{\pi}^e_j$$
$$\underline{K}^e_{ni} \underline{S}_i + \underline{K}^e_{nj} \underline{S}_j + \underline{K}^e_{nn} \underline{S}_n + \underline{q}^e_n = \underline{\pi}^e_n$$

→ 3 eq. vettori, equil. a tr. e 2 a rot.

(negli $\underline{S}$ non scrive la "e", sono rif. al corpo al quale l'elemento appartiene)

Equilibrio del nodo: reaz. = f. app. al nodo

Def. $\underline{F}_i = \begin{bmatrix} F_i \\ m_{1i} \\ m_{2i} \end{bmatrix}$; $\underline{F}_i = \sum_e \underline{\pi}^e_i$

Mancano le cond. essenziali:

$\underline{S}_{(i)} = \hat{\underline{S}}_{(i)} = \begin{bmatrix} \hat{\omega} \\ \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix}$ Sost. con $\underline{S}_{(i)}$.

Ma in th. Kirchhoff le c. essenz. sono assegn. $\hat{\omega}$ e $\hat{\omega}_{,n}$.

$$\omega_{,n} = \omega_{,1} n_1 + \omega_{,2} n_2 = -\varphi_2 n_1 + \varphi_1 n_2 = \hat{\omega}_{,n}$$

$$\omega_{,s} = \omega_{,1} t_1 + \omega_{,2} t_2 = -\omega_{,1} n_2 + \omega_{,2} n_1 =$$
$$= \varphi_2 n_2 + \varphi_1 n_1 = \hat{\omega}_{,s}$$

 Sost. le 3 eq. di equil. del nodo è

con $w = \hat{w}$

$$\varphi_1 n_2 - \varphi_2 n_1 = \hat{w}_{,n}$$
$$\varphi_1 n_1 + \varphi_2 n_2 = \hat{w}_{,s}$$

Se $n = 1,2$ mi ritrovo assegnate $\hat{w}, \hat{\varphi}_1, \hat{\varphi}_2$.

$\#$

TEORIA DI REISSNER - MINDLIN

Fibre rimangono rettilinee e non si estendono, ma possono variare l'angolo rip. a x_1, x_2, x_3, piano medio.

$\varepsilon_{33} = 0$ (no ext.)

Se fibra rimane rettilinea angoli non variano lungo essa: $\varepsilon_{31,3} = \varepsilon_{32,3} = 0$

Quindi:
- $u_{3,3} = 0 \rightarrow u_3 = w(x_1, x_2)$
- $(u_{3,1} + u_{1,3})_{,3} = 0 \rightarrow \cancel{u_{3,13}} + u_{1,33} = 0$. Int. e ho $u_{1,3} = \varphi_1(x_1, x_2)$. Int: $u_1 = \tilde{u}_1(x_1, x_2) + x_3 \varphi_1(x_1, x_2)$
- $u_2 = \tilde{u}(x_1, x_2) + x_3 \varphi_2(x_1, x_2)$
- $u_3 = w(x_1, x_2)$

Questa è anche la <u>teoria delle PIASTRE DEFORMABILI a TAGLIO</u>.

Dato il campo di spost., si ha il campo di def:

$$\varepsilon_{11} = u_{1,1} = \tilde{u}_{1,1} + x_3 \varphi_{1,1}$$
$$\varepsilon_{22} = u_{2,2} = \tilde{u}_{2,2} + x_3 \varphi_{2,2}$$
$$\varepsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = \frac{1}{2}(\tilde{u}_{1,2} + \tilde{u}_{2,1}) + x_3 \frac{1}{2}(\varphi_{1,2} + \varphi_{2,1})$$
$$\varepsilon_{31} = \frac{1}{2}(u_{1,3} + u_{3,1}) = \frac{1}{2}(\varphi_1 + w_{,1})$$
$$\varepsilon_{32} = \frac{1}{2}(u_{2,3} + u_{3,2}) = \frac{1}{2}(\varphi_2 + w_{,2})$$
$$\varepsilon_{33} = 0$$

Gli sforzi in mat. ort. romerico:

$$T_{11} = \frac{E_{(1)}}{1-\nu_{(12)}\nu_{(21)}}\left(\varepsilon_{11} + \nu_{(21)}\varepsilon_{22}\right)$$

$$T_{22} = \frac{E_{(2)}}{1-\nu_{(12)}\nu_{(21)}}\left(\varepsilon_{22} + \nu_{(12)}\varepsilon_{11}\right)$$

$$T_{12} = 2G_{(12)}\varepsilon_{12} \; ; \; T_{13} = 2G_{(13)}\varepsilon_{13}$$

Se trasv. isotropo:

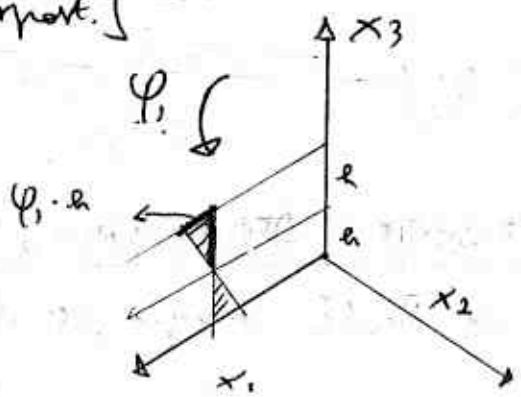
$$T_{11} = \frac{E}{1-\nu^2}\left(\varepsilon_{11} + \nu\varepsilon_{22}\right)$$

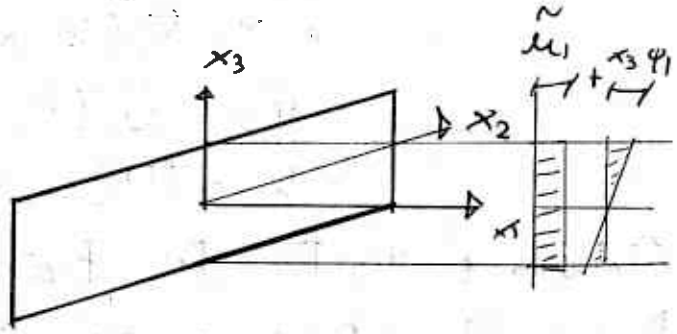
$$T_{22} = \frac{E}{1-\nu^2}\left(\varepsilon_{22} + \nu\varepsilon_{11}\right)$$

$$T_{12} = \frac{E}{1+\nu}\varepsilon_{12} \; ; \; T_{13} = 2\hat{\mu}\,\varepsilon_{13} \; ; \; T_{23} = 2\hat{\mu}\,\varepsilon_{23}$$

Gli sforzi risultanti sono: (tr. in.):

[sport.]

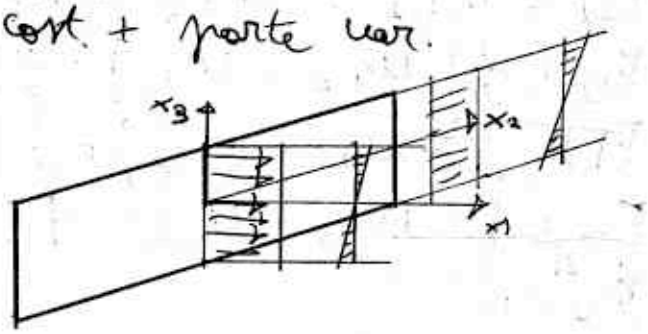




$$\varphi_2 = \frac{\varphi_1 \cdot h}{h} \qquad -\varphi_2 = \varphi_1$$

[tens.] Anche qui parte cost. + parte var.

Sforzi membr. + sf. tagliante (x_2) o mom. flett. (x_3)



[parte cont.]

$$N_{11} = \int_{-h}^{h} T_{11}\, dx_3 = \frac{2hE}{1-\nu^2}\left(\tilde{u}_{1,1} + \nu\, \tilde{u}_{2,2}\right)$$

$$N_{22} = \int_{-h}^{h} T_{22}\, dx_3 = \frac{2hE}{1-\nu^2}\left(\tilde{u}_{2,2} + \nu\, \tilde{u}_{1,1}\right)$$

$$N_{12} = \int_{-h}^{h} T_{12}\, dx_3 = \frac{2hE}{1+\nu}\,\frac{1}{2}\left(\tilde{u}_{1,2} + \tilde{u}_{2,1}\right)$$

[parte var.]

Si pone $D = \frac{E}{1-\nu^2}\int_{-h}^{h} x_3^2\, dx_3 = \frac{2h^3E}{3(1-\nu^2)}$, allora:

$$M_{11} = D\left(\varphi_{1,1} + \nu\, \varphi_{2,2}\right) = \int_{-h}^{h} T_{11}\, x_3\, dx_3$$

$$M_{22} = D\left(\varphi_{2,2} + \nu\, \varphi_{1,1}\right)$$

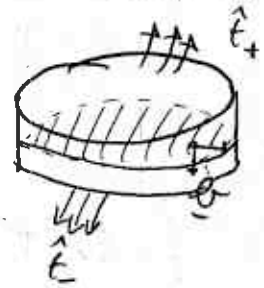
$$M_{12} = D(1-\nu)\,\frac{1}{2}\left(\varphi_{1,2} + \varphi_{2,1}\right)$$

$$Q_1 = N_{31} = \int_{-h}^{h} T_{31}\, dx_3 = 2h\hat{\mu}\left(\varphi_1 + w_{,1}\right)$$

$$Q_2 = N_{32} = 2h\hat{\mu}\left(\varphi_2 + w_{,2}\right)$$

Eq. equil. Da PLV in 3D, se si integra su spessore:

$$\int_N \Big(T_{\alpha\beta}\left(\tilde{v}_{\alpha,\beta} + x_3\, \chi_{\alpha,\beta}\right) + T_{\alpha 3}\left(\eta_\alpha + \chi_\alpha\right)\Big)\, dA =$$



↳ lav. interno

$$\left[\ \underline{u} = \left(\tilde{u}_\alpha + x_3\, \varphi_\alpha\right)\underline{e}_\alpha + w\, \underline{e}_3;\right.$$

$$\left.\ \underline{v} = \left(\tilde{v}_\alpha + x_3\, \chi_\alpha\right)\underline{e}_\alpha + \eta\, \underline{e}_3\ \right]$$

$$= \int_{\mathcal{N}_r} \left(b_\alpha \left(\tilde{u}_\alpha + x_3 \chi_\alpha \right) + b_3 \eta \right) dA + \int_{\partial \mathcal{N}_r} \left(t_\alpha \left(\tilde{u}_\alpha + x_3 \chi_\alpha \right) + t_3 \eta \right) d(\partial A)$$

↳ Lav. esterno

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$$\int_{\mathcal{N}_r} \left(T_{\alpha\beta} \left(\tilde{u}_{\alpha,\beta} + x_3 \chi_{\alpha,\beta} \right) + T_{3\alpha} \left(\eta_{,\alpha} + \chi_\alpha \right) \right) =$$

$$\int_{\mathcal{N}_r} \left(b_\alpha \left(\tilde{u}_\alpha + x_3 \chi_\alpha \right) + b_3 \eta \right) + \int_{\partial \mathcal{N}_r} \left(\hat{t}_\alpha \left(\tilde{u}_\alpha + x_3 \chi_\alpha \right) + \hat{t}_3 \eta \right)$$

$$\int_r = \int_{\mathcal{N}} \int_{-h}^{h} \; ; \text{ quindi:}$$

$$\int_{\mathcal{N}} \left(N_{\alpha\beta} \tilde{u}_{\alpha,\beta} + M_{\alpha\beta} \chi_{\alpha,\beta} + Q_\alpha \left(\eta_{,\alpha} + \chi_\alpha \right) \right) =$$

[contorno diviso in 3 parti, sup, inf e mant.]

$$\int_{\mathcal{N}} \left(q_\alpha \tilde{u}_\alpha + s_\alpha \chi_\alpha + q_3 \eta \right) + \int_{\partial \mathcal{N}} \left(\hat{f}_\alpha \tilde{u}_\alpha + \hat{c}_\alpha \chi_\alpha + \hat{f}_3 \eta \right) \text{ con:}$$

$$q_\alpha = \int_{-h}^{h} b_\alpha \, dx_3 + \hat{t}_\alpha^+ + \hat{t}_\alpha^-$$

$$s_\alpha = \int_{-h}^{h} x_3 b_\alpha \, dx_3 + h \left(\hat{t}_\alpha^+ - \hat{t}_\alpha^- \right)$$

$$\hat{f}_i = \int_{-h}^{h} \hat{t}_i \, dx_3 \quad (\text{su } \partial \mathcal{N}) \; ; \quad \hat{c}_\alpha = - \int_{-h}^{h} \hat{t}_\alpha x_3 \, dx_3 \quad (\text{su } \partial \mathcal{N})$$

147 Ora int. per parti!

$$-\int_{\gamma}\left[\left(N_{\alpha\beta,\beta}+q_\alpha\right)\tilde{v}_\alpha+\left(M_{\alpha\beta,\beta}+\mathcal{C}_\alpha-Q_\alpha\right)\chi_\alpha\right]-\int_{\gamma}\left(Q_{\alpha,\alpha}+q_3\right)\eta+$$

$$\left[N_{\alpha\beta}\tilde{v}_{\alpha,\beta}=\left(N_{\alpha\beta}\tilde{v}_\alpha\right)_{,\beta}-N_{\alpha\beta,\beta}\tilde{v}_\alpha\right]$$

$$+\int_{\partial\gamma}\left(\left(N_{\alpha\beta}n_\beta-\hat{f}_\alpha\right)\tilde{v}_\alpha+\left(M_{\alpha\beta}n_\beta-\hat{c}_\alpha\right)\chi_\alpha+\left(Q_\alpha n_\alpha-\hat{f}_3\right)\eta\right)=0$$

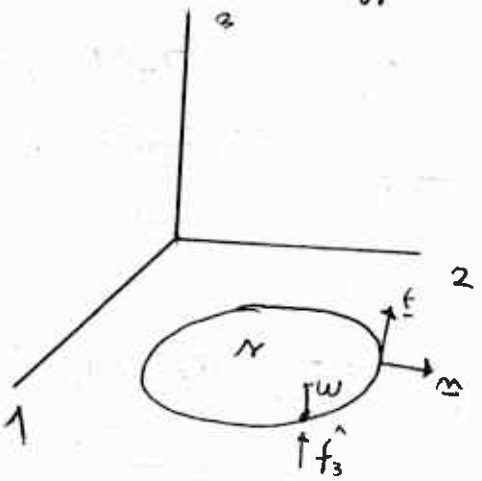
$$\left[Q_\alpha\eta_{,\alpha}=\left(Q_\alpha\eta\right)_{,\alpha}-Q_{\alpha,\alpha}\eta\right]$$

Valide $\forall$ variazione:

$$\begin{cases} N_{\alpha\beta,\beta}+q_\alpha=0 \\ Q_{\alpha,\alpha}+q_3=0 \\ M_{\alpha\beta,\beta}+\mathcal{C}_\alpha-Q_\alpha=0 \end{cases} \text{ in } \gamma$$

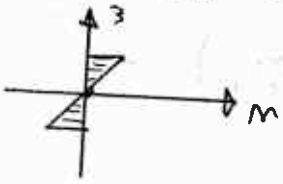
$$\begin{cases} \tilde{u}_\alpha=\hat{u}_\alpha \text{ oppure } N_{\alpha\beta}n_\beta=\hat{f}_\alpha \\ w=\hat{w} \text{ oppure } Q_\alpha n_\alpha=\hat{f}_3 \\ \psi_\alpha=\hat{\varphi}_\alpha \text{ oppure } M_{\alpha\beta}n_\beta=\hat{c}_\alpha \end{cases}$$

Regime membranale descritto da stesse eq. nelle ip. di Kirchhoff.



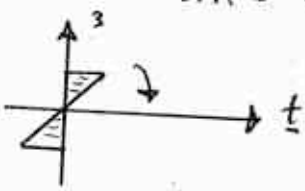
Se assegno ψ_n, assegno $u_n = x_3\,\varphi_n$ ($\underline{n}$ ha sgn. fisico + preciso)

A ψ_n corrisp. $u_t = x_3\,\psi_t$



fa ruotare la fibra intorno a $\underline{t}$

Se $\psi_n > 0$, rot. porta 3 su 1. Quindi σ assegno rot. intorno a $\underline{t}$ σ assegno mom. flettente.



Questa (che) è opposto rot. intorno a $\underline{t}$ (da 3 su 2, negativa). O assegno rot. tors. σ assegno $\hat{c}_\alpha$, coppia torsionale.

In componenti:

$$2h\hat{\mu}(w_{,1}+\varphi_1)_{,1} + 2h\hat{\mu}(w_{,2}+\varphi_2)_{,2} + q = 0$$

$$\begin{bmatrix} M_{11,1} + M_{12,2} + \alpha_1 - Q_1 = 0 \\ M_{21,1} + M_{22,2} + \alpha_2 - Q_2 = 0 \end{bmatrix}$$

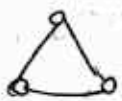
$$D\left(\varphi_{1,1} + \nu\varphi_{2,2}\right)_{,1} + \frac{D(1-\nu)}{2}\left(\varphi_{1,2}+\varphi_{2,1}\right)_{,2} + \alpha_1 = 2h\hat{\mu}(w_{,1}+\varphi_1)$$

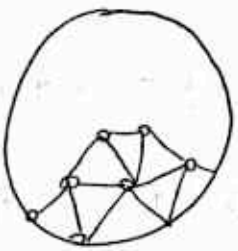
$$\frac{D(1-\nu)}{2}\left(\varphi_{1,2}+\varphi_{2,1}\right)_{,1} + D\left(\varphi_{1,2}+\nu\varphi_{1,1}\right)_{,2} + \alpha_2 = 2h\hat{\mu}(w_{,2}+\varphi_2)$$

Studio agli elementi finiti:

Fix. corpo suddiviso in elem. triang.

Dobbiamo scrivere nella form. variaz.:





$$\begin{cases} M_{\alpha\beta,\beta} + \alpha_\alpha - Q_\alpha = 0 \\ Q_{\alpha,\alpha} + q = 0 \end{cases}$$

χ_α $\quad \varphi_\alpha = \hat{\varphi}_\alpha$ su $\partial\Omega_1$ | $M_{\alpha\beta}n_\beta = \hat{c}_\alpha$ su $\partial\Omega_2$

η $\quad w = \hat{w}$ su $\partial\Omega_1$ | $Q_\alpha n_\alpha = \hat{f}_3$ su $\partial\Omega_2$

Molt. per variaz. e int su Ω:

$$\int_\Omega \chi_\alpha\left(M_{\alpha\beta,\beta} + \alpha_\alpha - Q_\alpha\right) = 0$$

Dopo. th. diverg.:

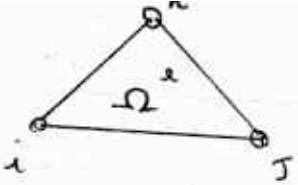
$$\int_\Omega \eta\left(Q_{\alpha,\alpha} + q\right) = 0$$

$$0 = \int_\Omega \left(-M_{\alpha\beta}\chi_{\alpha,\beta} - Q_\alpha\eta_{,\alpha} + \alpha_\alpha\chi_\alpha - Q_\alpha\chi_\alpha + q\eta\right) +$$

$$+ \int_{\partial\Omega_2}\left(M_{\alpha\beta}n_\beta\chi_\alpha + Q_\alpha n_\alpha\eta\right)$$

sostituisco:

$$\int_{\partial\Omega_2}\left(\hat{c}_\alpha\chi_\alpha + \hat{f}_3\eta\right)$$



Cons. azioni nel bordo pari a quelle esercitate da altri elementi. (p. esterne pas. ridotte a f. nodali)

$$\underline{S}(i) = \begin{bmatrix} S(i)1 \\ S(i)2 \\ S(i)3 \end{bmatrix} = \begin{bmatrix} w_1(i) \\ \varphi_1(i) \\ \varphi_2(i) \end{bmatrix} \quad ; \quad \underline{u}^e = \begin{bmatrix} w \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = \underline{\Phi}^e \, \underline{S}^e$$

Assumiamo per componenti di $\underline{\Phi}^e$ polinomi lineari:

- $w^e(x_1, x_2) = S(i)1 \; \phi_i^e(x_1, x_2) + S(j)1 \; \phi_j^e(x_1, x_2) + S(k)1 \; \phi_k^e(x_1, x_2)$
- $\varphi_1^e(x_1, x_2) = S(i)2 \; \Theta_i^e(x_1, x_2) + S(j)2 \; \Theta_j^e(x_1, x_2) + S(k)2 \; \Theta_k^e(x_1, x_2)$
- $\varphi_2^e(x_1, x_2) = S(i)3 \; w_i^e(x_1, x_2) + S(j)3 \; w_j^e(x_1, x_2) + S(k)3 \; w_k^e(x_1, x_2)$

con:

$$\phi_m^e = a_0 m + a_1 m x_1 + a_2 m x_2$$

$$\Theta_m^e = b_0 m + b_1 m x_1 + b_2 m x_2$$

$$w_m^e = c_0 m + c_1 m x_1 + c_2 m x_2$$

C. al contorno per det. i coeff.:

$$\phi_i^e(x_{(i)1}, x_{(i)2}) = 1 \; ; \; \phi_i^e(x_{(j)1}, x_{(j)2}) = 0 \; ; \; \phi_i^e(x_{(k)1}, x_{(k)2}) = 0$$

$$\underline{\Phi}^e = \begin{bmatrix} \phi_i^e & 0 & 0 & \phi_j^e & 0 & 0 & \phi_k^e & 0 & 0 \\ 0 & \Theta_i^e & 0 & 0 & \Theta_j^e & 0 & 0 & \Theta_k^e & 0 \\ 0 & 0 & w_i^e & 0 & 0 & w_j^e & 0 & 0 & w_k^e \end{bmatrix} \cdot \begin{bmatrix} S(i)1 \\ S(i)2 \\ S(i)3 \\ \vdots \\ S(k)3 \end{bmatrix}$$

Variazione: $\underline{v}^e = \underline{\Phi}^e \underline{p}^e$.

Misura di deformazione: $\underline{\varepsilon}^e = \begin{bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} =$

$$= \begin{bmatrix} \frac{1}{2}\frac{\partial}{\partial x_1} & \frac{1}{2} & 0 \\ \frac{1}{2}\frac{\partial}{\partial x_2} & 0 & \frac{1}{2} \\ 0 & \frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_2} \\ 0 & \frac{1}{2}\frac{\partial}{\partial x_2} & \frac{1}{2}\frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} w^e \\ \varphi_1^e \\ \varphi_2^e \end{bmatrix} = \underline{D}\,\underline{u}^e = \left(\underline{D}\,\underline{\Phi}^e\right)\underline{s}^e = \underline{B}^e \underline{s}^e$$

Misura di sforzo:

$$\underline{\sigma}^e = \begin{bmatrix} 2Q_1 \\ 2Q_2 \\ M_{11} \\ M_{22} \\ 2M_{12} \end{bmatrix} = \begin{bmatrix} gh\hat{\mu} & 0 & 0 & 0 & 0 \\ 0 & gh\hat{\mu} & 0 & 0 & 0 \\ 0 & 0 & D & D\nu & 0 \\ 0 & 0 & D\nu & D & 0 \\ 0 & 0 & 0 & 0 & 2D(1-\nu) \end{bmatrix} \underline{\varepsilon}^e = \underline{C}\,\underline{B}^e \underline{s}^e$$

(tagliante, $\frac{1}{2}$)

Abbiamo:

$$0 = \int_{\Omega^e} \left(M_{\alpha\beta} \chi_{\alpha,\beta} + Q_\alpha (\eta_{,\alpha} + \chi_\alpha) \right) - \int_{\Omega^e} \left(\alpha_\alpha \chi_\alpha + q\eta \right) - \int_{\partial\Omega^e} \left(\hat{C}_\alpha \chi_\alpha + \hat{f}_3 \eta \right)$$

$$0 = \int_{\Omega^e} \underline{C}\,\underline{B}^e \underline{s}^e \cdot \underline{B}^e \underline{p}^e - \int_{\Omega^e} \underline{p}^e \cdot \underline{\Phi}^e \underline{p}^e - \int_{\partial\Omega^e} \hat{\underline{f}}^e \cdot \underline{\Phi}^e \underline{p}^e \quad \text{con}$$

$$\underline{v}^e = \begin{bmatrix} \eta \\ \chi_1 \\ \chi_2 \end{bmatrix}; \quad \underline{p}^e = \begin{bmatrix} q \\ \alpha_1 \\ \alpha_2 \end{bmatrix}; \quad \underline{f}^e = \begin{bmatrix} f_3 \\ C_1 \\ C_2 \end{bmatrix}$$

(15) allora

$$0 = \underbrace{\left(\int_{\Omega^e} \underline{B}^{e^T} \underline{C}\, \underline{B}\right)}_{\underline{k}^e} \underline{s}^e \cdot \underline{\beta}^e - \underbrace{\left(\int_{\Omega^e} \underline{\Phi}^{e^T} \underline{p}^e\right)}_{\underline{q}^e} \cdot \underline{\beta}^e - \underbrace{\left(\int_{\partial\Omega^e} \underline{\Phi}^{e^T} \underline{f}^e\right)}_{\underline{r}^e} \cdot \underline{\beta}^e$$

$$\left(\underline{k}^e \underline{s}^e + \underline{q}^e - \underline{r}^e\right) \cdot \underline{\beta}^e = 0 \quad \forall\, \underline{\beta}^e \quad \text{e quindi}$$

$$\boxed{\underline{K}^e \underline{s}^e + \underline{q}^e = \underline{r}^e} \quad \text{9 eq} \quad \forall \text{ nodo:}$$

$$\underline{k}_{ii}^e \underline{s}_i + \underline{k}_{ij}^e \underline{s}_j + \underline{k}_{ik}^e \underline{s}_k + \underline{q}_i^e = \underline{r}_i^e$$

Se $\underline{F}_i = \begin{bmatrix} F_3 \\ m_1 \\ m_2 \end{bmatrix}$ (forze esterne agg. al nodo),

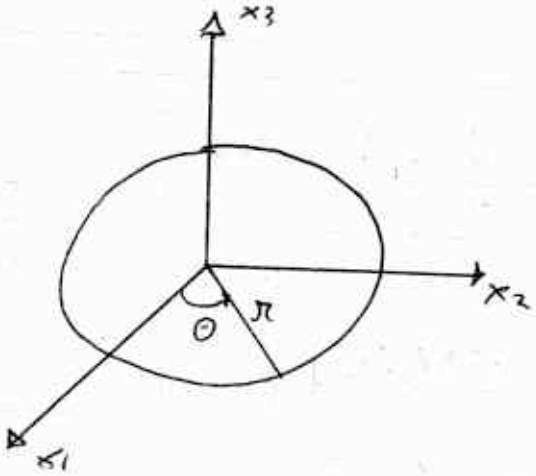
$$\underline{F}_i = \sum_e \underline{r}_i^e = \left(\sum k_{ii}^e\right) \underline{s}_i + \sum_e \left(\underline{k}_{ij}^e \underline{s}_j + \underline{k}_{ik}^e \underline{s}_k\right) + \sum_e \underline{q}_i^e$$

Se i è nodo di bordo con c. essenz. si sopprime e la si sostituisce con $\underline{s}_i = \hat{\underline{s}}_i$

Fatte le prime scelte sui nodi e le f. di forma, metodo FEM è "automatico".

H

PIASTRE CIRCOLARI NELLA TEORIA DI REISSNER-MINDLIN



- $\varepsilon_{zz} = 0$; $u_{z,z} = 0$, $u_z = w(r,\theta)$
- $\varepsilon_{zr,z} = 0$; $(u_{r,z} + \cancel{u_{z,r}})_{,z} = 0$

  quindi $u_{r,zz} = 0$, $u_r =$

  $= \tilde{u}_r(r,\theta) + z\,\varphi_r(r,\theta)$
- $\varepsilon_{z\theta,z} = 0$; $(u_{\theta,z} + u_{z,\theta})_{,z} = 0$,

  $u_{\theta,zz} = 0$, $u_\theta =$

$\tilde{u}_\theta(r,\theta) + z\,\varphi_\theta(r,\theta)$.

Sym assiale, quindi $u_\theta = 0$ e altre fun. $\not\theta$.

$$\underline{u} = \left(u(r) + z\,\varphi(r)\right)\underline{e}_r(\theta) + w(r)\,\underline{e}_z$$

$$\varepsilon_{rr} = u_{,r} + z\,\varphi_{,r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left(u + z\,\varphi\right)$$

$$\varepsilon_{rz} = \frac{1}{2}\left(w_{,r} + \varphi\right)$$

Per mat. trasv. isotropo:

$$T_{rr} = \frac{E}{1-\nu^2}\left(u_{,r} + \frac{\nu}{r}u + z\left(\varphi_{,r} + \frac{\nu}{r}\varphi\right)\right)$$

(da $T_{rr} = \frac{E}{1-\nu^2}\left(\varepsilon_{rr} + \nu\,\varepsilon_{\theta\theta}\right)$)

$$T_{\theta\theta} = \frac{E}{1-\nu^2}\left(\frac{1}{r}u + \nu u_{,r} + z\left(\frac{1}{r}\varphi + \nu\,\varphi_{,r}\right)\right)$$

(da $T_{\theta\theta} = \frac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\,\varepsilon_{rr}\right)$)

$$T_{zr} = \hat{\mu}\left(w_{,r} + \varphi\right) \quad \left(\text{da } T_{zr} = 2\hat{\mu}\,\varepsilon_{rz}\right)$$

altri nulli.

Sforzi risultanti:

$$N_{rr} = \int_{-h}^{h} T_{rr}\,dz = \frac{2hE}{1-\nu^2}\left(u_{,r} + \frac{\nu}{r}u\right)$$

$$N_{\theta\theta} = \int_{-h}^{h} T_{\theta\theta}\,dz = \frac{2hE}{1-\nu^2}\left(\frac{1}{r}u + \nu u_{,r}\right)$$

(153) Essendo $q_r = \int_{-h}^{h} b_r\,dz + \hat{t}_r^{+} + \hat{t}_r^{-}$,

$N_{rr,r} + \frac{1}{r}(N_{rr} - N_{\theta\theta}) + q_r = 0$. Sostituendo:

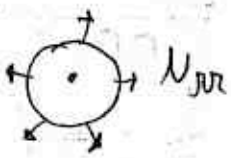
$$\frac{2hE}{1-\nu^2}\left(u_{,rr} - \cancel{\frac{\nu}{r^2}u} + \cancel{\frac{\nu}{r}u_{,r}} + \frac{1}{r}u_{,r} + \cancel{\frac{\nu}{r^2}u} - \cancel{\frac{\nu}{r}u_{,r}}\right) + q_r = 0$$

$$\frac{2hE}{1-\nu^2}\left(u_{,rr} + \left(\frac{1}{r}u\right)_{,r} + q_r\right) = 0$$

$$\frac{2hE}{1-\nu^2}\left(u_{,r} + \frac{1}{r}u\right)_{,r} + q_r = 0$$

$$\boxed{\frac{2hE}{1-\nu^2}\left(\frac{1}{r}(ru)_{,r}\right)_{,r} + q_r = 0} \rightarrow$$ eq. diff. membrana a sym assiale

Disegno u su ∂r e $N_{\partial r} = \hat{f}_r$ (f. per unità di lung. sul bordo in dir r)



$$M_{rr} = \int_{-h}^{h} T_{rr}\, z\, dz \quad \text{quindi}$$

$$M_{rr} = D\left(\varphi_{,r} + \frac{\nu}{r}\varphi\right); \quad M_{\theta\theta} = D\left(\frac{1}{r}\varphi + \nu\varphi_{,r}\right);$$

$$Q_r = N_{zr} = 2h\hat{\mu}\left(w_{,r} + \varphi\right)$$

$$N_{zr,r} + \frac{1}{r}N_{zr} + q_{(z)} = 0; \quad \frac{1}{r}(rN_{zr})_{,r} = -q \quad ①$$

$$M_{rr,r} + \frac{1}{r}(M_{rr} - M_{\theta\theta}) - N_{zr} = 0 \quad . \text{ Sost:}$$

$$D\left(\varphi_{,rr} - \cancel{\frac{\nu}{r^2}\varphi} + \cancel{\frac{\nu}{r}\varphi_{,r}} + \frac{1}{r}\varphi_{,r} + \cancel{\frac{\nu}{r^2}\varphi} - \frac{1}{r^2}\varphi - \cancel{\frac{\nu}{r}\varphi_{,r}}\right) = N_{zr}$$

$D\left(\varphi_{,r} + \frac{1}{r}\varphi\right)_{,r}$ e quindi $D\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r} = N_{zr}$

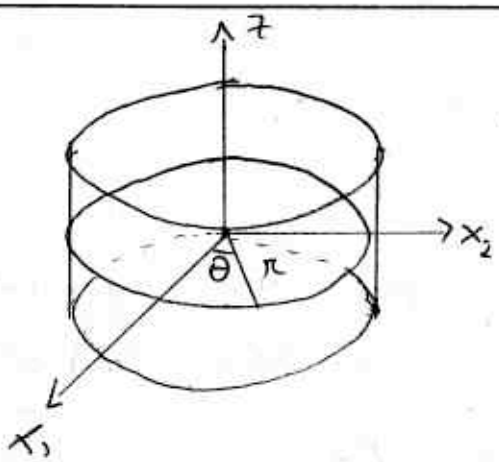
Sost. in ①: $\frac{1}{r}\left(r\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r}\right)_{,r} = -\frac{q}{D}$ → eq. per φ

$$w_{,r} = \frac{D}{2h\hat{\mu}}\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r} - \varphi$$

C. al cont: assegno o w o f oppure φ o M_{rr}

f: trasl. al piano medio — M: flett.

25/5/09



$$N_{zr,r} + \frac{1}{r}N_{zr} + q = 0$$

$$M_{rr,r} + \frac{1}{r}(M_{rr} - M_{\theta\theta}) + \cancel{m} - N_{zr} = 0$$

$$N_{zr} = 2h\hat{\mu}(w_{,r} + \varphi)$$

$$M_{rr} = D\left(\varphi_{,r} + \frac{\nu}{r}\varphi\right)$$

$$M_{\theta\theta} = D\left(\frac{1}{2}\varphi + \nu\varphi_{,r}\right)$$

$$\frac{1}{r}\left(r\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r}\right)_{,r} = -\frac{q}{D}$$

$$w_{,r} = \frac{D}{2h\hat{\mu}}\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r} - \varphi$$

$$\left(r\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r}\right)_{,r} = 0 \; ; \; r\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r} = a_1$$

$$\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r} = \frac{a_1}{r} \; ; \; \frac{1}{r}(r\varphi)_{,r} = a_1 \ln r + a_2$$

$$(r\varphi)_{,r} = a_1 r \ln r + a_2 r \; ; \; r\varphi = \frac{a_1}{4}r^2(2\ln r - 1) + a_2\frac{r^2}{2} + a_3$$

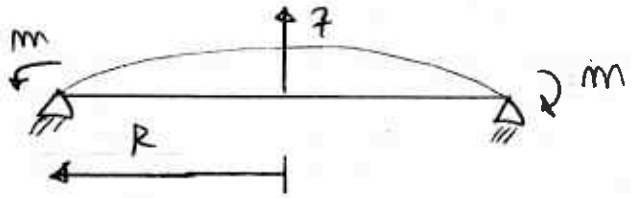
$$\varphi = \frac{a_1}{4}r(2\ln r - 1) + \frac{a_2}{2}r + \cancel{\frac{a_3}{r}}$$

155 $$\varphi_{,r} = \frac{a_1}{4}(2\ln r + 1) + \frac{a_2}{2}$$

↳ per avere mom. finiti.

$$\varphi = \frac{a_2}{2} r$$

Ex: Piastra appoggiata soggetta a distr. unif. di coppie flettenti nel bordo



$$w(R) = 0$$

$$M_{rr}(R) = m = D\frac{a_2}{2}(1+\nu)$$

$$M_{rr} = D\left(\varphi_{,r} + \frac{\nu}{r}\varphi\right) = D\frac{a_2}{2}(1+\nu)$$

$$\frac{a_2}{2} = \frac{m}{D(1+\nu)} \; ; \; \varphi = \frac{m}{D(1+\nu)} r$$

$$\left(\frac{m}{D(1+\nu)} r^2\right)_{,r} = \frac{2m\cancel{r}}{D(1+\nu)}$$ quindi nullo.

$$M_{rr} = m \; ; \; M_{\theta\theta} = D\left(\frac{1}{2}\varphi + \nu\varphi_{,r}\right)$$

$$M_{rr} = M_{\theta\theta}$$

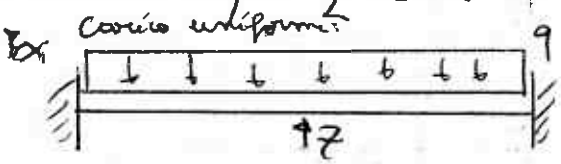
$$2\hat{\mu}(w_{,r} + \varphi) = 0 \Rightarrow w_{,r} = -\varphi$$

$$w_{,r} = -\frac{m}{D(1+\nu)} r \; ; \; w = -\frac{m}{D(1+\nu)}\frac{r^2}{2}$$

$$C = \frac{mR^2}{2D(1+\nu)}$$ allora $$w = \frac{m}{2(1+\nu)D}(R^2 - r^2)$$

$$T_{zr} = 2\hat{\mu}\frac{1}{2}\varepsilon_{zr} = 0$$

Ex carico uniforme



Ci vuole l'int. part.:

$$\varphi = -\frac{qr^3}{16D} \; ; \; r\cdot\varphi_{,r} = \frac{qr^3}{4D} \; ; \; \frac{1}{r}\cdot\varphi_{,r} = \frac{qr^2}{4D} \; ; \; \frac{qr}{D} \; ; \; \frac{q}{D}$$

Quindi la sol. è:

$$\varphi = \frac{a_2}{2} r - \frac{q r^3}{16 D}$$. Le cond. sono $w(R)=0$ e $\varphi(R)=0$

$$\frac{a_2}{2} = \frac{q R^2}{16 D} \quad ; \quad \varphi = \frac{q}{16 D}\left(R^2 r - r^3\right)$$

$$w_{,r} = \frac{D}{2h\hat{\mu}}\left(\frac{1}{r}(r\varphi)_{,r}\right)_{,r} - \varphi$$

$$\varphi_{,r} = \frac{q}{16D}\left(R^2 - 3r^2\right)$$

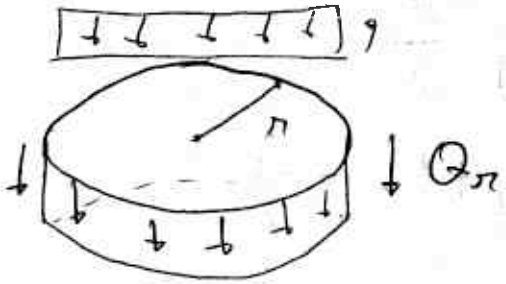
$$M_{rr} = \frac{q}{16}\left(R^2 - 3r^2 + \nu\left(R^2 - r^2\right)\right) = \frac{q}{16}\left(R^2(1+\nu) - r^2(\nu+3)\right)$$

$$M_{\theta\theta} = \frac{q}{16}\left(R^2 - r^2 + \nu\left(R^2 - 3r^2\right)\right) = \frac{q}{16}\left(R^2(1+\nu) - r^2(1+3\nu)\right)$$

$$r\varphi = \frac{q}{16D}\left(R^2 r^2 - r^4\right) \quad ;$$

$$\frac{1}{r}(r\varphi)_{,r} = \frac{q}{16D}\left(2R^2 \cancel{r} - 4r^{\cancel{3}}\right) \quad ; \quad \text{der. risp. a } r: -\frac{q}{2D} r$$

$$N_{zr} = Q_r = 2h\hat{\mu}\left(w_{,r} + \varphi\right) = -\frac{q r}{2}$$



Infatti!

$$Q_2\, 2\pi r + q \pi r^2 = 0$$

$$\Downarrow$$

$$Q_r = -\frac{q r}{2}$$

$$w_{,r} = -\frac{q}{4h\hat{\mu}} r - \frac{q}{16D}\left(R^2 r - r^3\right)$$

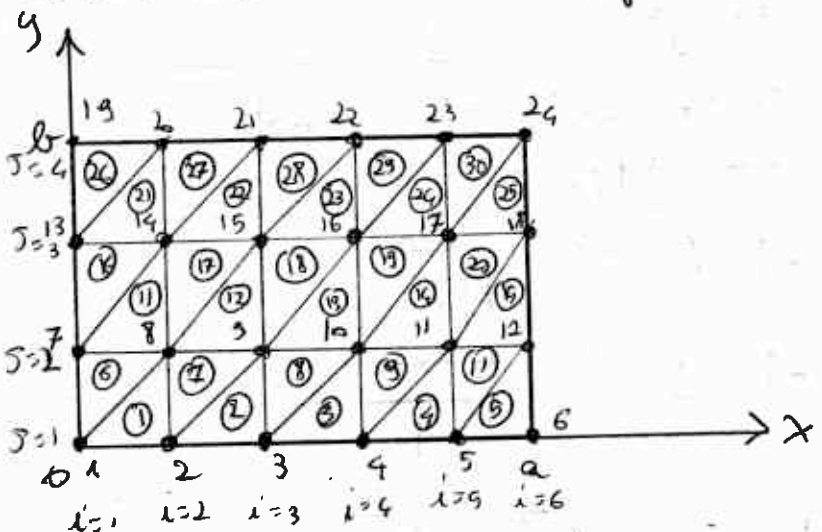
$$w = -\frac{q}{8h\mu} r^2 - \frac{q}{16D}\left(R^2 \frac{r^2}{2} - \frac{r^4}{4}\right) + C$$

$$C = \frac{q R^2}{8h\hat{\mu}} + \frac{q}{64D} R^4$$

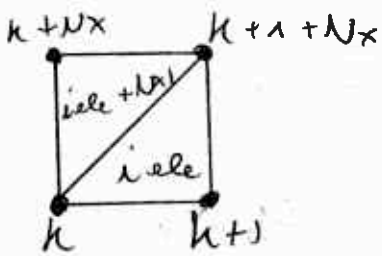
$$w = \frac{q}{8h\mu}\left(R^2 - r^2\right) + \frac{q}{64D}\left(R^2 - r^2\right)^2$$

(torna al FEM per Nuovo ref. membranale)

Cons. elementi triangolari



Suddiv. in parti uguali (3 x 15) e poi in triangoli. Numeriamo poi i nodi. Num. poi i triangoli di sotto e quelli di sopra. i = col.; J = righe



x lun: lunghezza lato su asse x

y lun: " " // a y

N_x: nodi su x = 6

N_y: " " y = 4

$\Big\}\ N_m = N_x \cdot N_y$

Qui: $N_{x1} = N_x - 1 = 5$; $N_{y1} = N_y - 1 = 3$; $N_e = 2 N_{x1} \cdot N_{y1}$

$$\begin{cases} k = i + (J-1)\cdot N_x \\ i\,ele = i + 2(J-1)\cdot N_{x1} \end{cases}$$

Per ogni nodo, quindi:

$i = 1, \dots, N_{x1}$

$J = 1, \dots, N_{y1}$

Matrice nodale $(N_e, 3)$ [con $\forall$ elem i suoi 3 nodi].

Poniamo

$$\text{nodiele}(\text{iele}, 1) = k$$
$$\text{nodiele}(\text{iele}, 2) = k + 1$$
$$\text{nodiele}(\text{iele}, 3) = k + 1 + N_x$$
$$\text{nodiele}(\text{iele} + N_{x1}, 1) = k$$
$$\text{nodiele}(\text{iele} + N_{x1}, 2) = k + 1 + N_x$$
$$\text{nodiele}(\text{iele} + N_{x1}, 3) = k + N_x$$

→ numero i nodi in senso antiorario

Ci serve conservare le coordinate dei nodi!

$$i\ \text{nodo} = i + (j-1) N_x$$
$$x\ \text{nodo}(i\text{nodo}) = (i-1) * \frac{x_{lun}}{N_{x1}}$$
$$y\ \text{nodo}(i\text{nodo}) = (j-1) * \frac{y_{lun}}{N_{y1}}$$

Dalla geom. si ha:

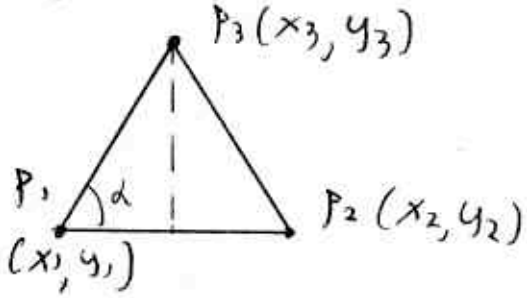
$$\det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \det \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

Dim:

$$x_2 y_3 - y_2 x_3 + y_1 x_3 - x_1 y_3 + x_1 y_2 - y_1 x_2 =$$
$$= x_2 (y_3 - y_1) + x_3 (y_1 - y_2) + x_1 (y_2 - y_3) \quad (+)$$

noi sommo e sottraggo

$$x_1 (y_3 - y_1) = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) \quad (U)$$

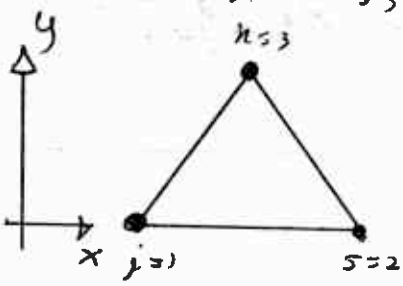


Calcoliamo area del tr.

$$\text{area}(P_1 P_2 P_3) = \frac{1}{2} |P_1 P_2| |P_1 P_3| \sin\alpha =$$
$$= \frac{1}{2} \overrightarrow{P_1 P_2} \times \overrightarrow{P_1 P_3} \cdot \underline{e}_z =$$

$$= \frac{1}{2} \det \begin{vmatrix} 0 & 0 & 1 \\ (x_2 - x_1) & (y_2 - y_1) & 0 \\ (x_3 - x_1) & (y_3 - y_1) & 0 \end{vmatrix} = \frac{1}{2} \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2\,\text{area}(P_1 P_2 P_3)$$



$$u = u_{(i)}\,\phi_i(x,y) + u_{(j)}\,\phi_j(x,y) + u_{(k)}\,\phi_k(x,y)$$

$$v = v_{(i)}\,\varphi_i(x,y) + v_{(j)}\,\varphi_j(x,y) + v_{(k)}\,\varphi_k(x,y)$$

ϕ e φ sono uguali (pol. di 1° grado e i coeff. si det. con le stesse condizioni)

$$\phi_i(x,y) = a_1 + b_1 x + c_1 y$$

$$\phi_j(x,y) = a_2 + b_2 x + c_2 y$$

$$\phi_k(x,y) = a_3 + b_3 x + c_3 y$$

Si ha:

$\phi_i(x_i, y_i) = 1$; $\phi_i(x_j, y_j) = 0$; $\phi_i(x_k, y_k) = 0$, cioè:

$$\begin{cases} a_1 + b_1 x_i + c_1 y_i = 1 \\ a_1 + b_1 x_j + c_1 y_j = 0 \\ a_1 + b_1 x_k + c_1 y_k = 0 \end{cases}$$

. Vogliamo a_1, b_1, c_1 ...

$$\det \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 2\Delta$$

area triangolo i, j, k

$$a_1 = \frac{1}{2\Delta} \det \begin{vmatrix} 1 & x_i & y_i \\ 0 & x_j & y_j \\ 0 & x_k & y_k \end{vmatrix} = \frac{1}{2\Delta}\left(x_j y_k - x_k y_j\right)$$

$$b_1 = \frac{1}{2\Delta} \det \begin{vmatrix} 1 & 1 & y_i \\ 1 & 0 & y_j \\ 1 & 0 & y_k \end{vmatrix} = \frac{1}{2\Delta}\left(y_j - y_k\right)$$

$$c_1 = \frac{1}{2\Delta} \det \begin{vmatrix} 1 & x_i & 1 \\ 1 & x_j & 0 \\ 1 & x_k & 0 \end{vmatrix} = \frac{1}{2\Delta}(x_k - x_j)$$

$\phi_j(x_i, y_i) = 0$; $\phi_j(x_j, y_j) = 1$; $\phi_j(x_k, y_k) = 0$

$\phi_j = a_2 + b_2 x + c_2 y$ funkce

$$\begin{cases} a_2 + b_2 x_i + c_2 y_i = 0 \\ a_2 + b_2 x_j + c_2 y_j = 1 \\ a_2 + b_2 x_k + c_2 y_k = 0 \end{cases}$$

analogicky a přímka!

$$a_2 = \frac{1}{2\Delta}(x_k y_i - x_i y_k)$$

$$b_2 = \frac{1}{2\Delta}(y_k - y_i)$$

$$c_2 = \frac{1}{2\Delta}(x_i - x_k)$$

Stejně pro ϕ_k:

$\phi_k(x_i, y_i) = 0$; $\phi_k(x_j, y_j) = 0$; $\phi_k(x_k, y_k) = 1$

$\phi_k = a_3 + b_3 x + c_3 y$

$$\begin{cases} a_3 + b_3 x_i + c_3 y_i = 0 \\ a_3 + b_3 x_j + c_3 y_j = 0 \\ a_3 + b_3 x_k + c_3 y_k = 0 \end{cases}$$

$$a_3 = \frac{1}{2\Delta}(x_i y_j - x_j y_i)$$

$$b_3 = \frac{1}{2\Delta}(y_i - y_j)$$

 $c_3 = \frac{1}{2\Delta}(x_j - x_i)$

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$$u(x,y) = u_{(i)}\phi_i + u_{(j)}\phi_j + u_{(k)}\phi_{(k)} =$$

$$= u_{(i)}(a_1 + b_1 x + c_1 y) + u_{(j)}(a_2 + b_2 x + c_2 y) +$$

$$+ u_{(k)}(a_3 + b_3 x + c_3 y)$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \phi_i & 0 & \phi_j & 0 & \phi_k & 0 \\ 0 & \psi_i & 0 & \psi_j & 0 & \psi_k \end{bmatrix} \begin{bmatrix} u_{(i)} \\ v_{(i)} \\ \vdots \\ \vdots \end{bmatrix}$$

$$\underline{\varepsilon}^e = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \frac{1}{2}\frac{\partial}{\partial y} & \frac{1}{2}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \qquad \frac{\partial \phi_i}{\partial x} = b_1 \qquad \frac{\partial \phi_i}{\partial y} = c_1$$

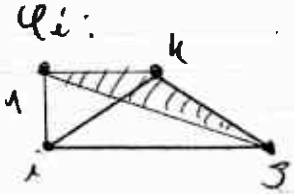
$$= \underline{D}\,\underline{u} = \underline{D}\,\underline{\Phi}\,\underline{s}^e = \underline{B}\,\underline{s}^e$$ → Vedi foglio 6

$$\underline{\sigma}^e = \begin{bmatrix} N_{11} \\ N_{22} \\ 2N_{12} \end{bmatrix} = \frac{Eh}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

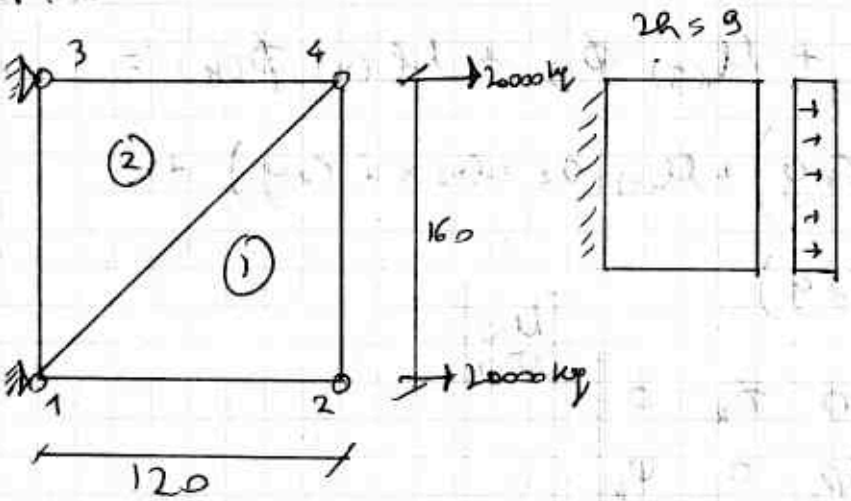
$$\underline{K}^e = \int_{\Omega^e} \underline{B}^T \underline{C}\,\underline{B} = \Delta B^T C B$$

Carichi: $$\int_{\Omega^e} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \cdot \underline{\Phi}\,\underline{f}^e = \int_{\Omega^e} \underline{\Phi}^T \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \int_{\Omega^e} \begin{bmatrix} \phi_i b_x \\ \psi_i b_y \\ \phi_j b_x \\ \psi_j b_y \\ \phi_k b_x \\ \psi_k b_y \end{bmatrix}$$ → GG[i=1] (prop. 4) GG[6]

φ_i:



Ex:



$E = 300\,GPa$

$\nu = 0{,}25$

FINE CORSO (26/5/2009)